The geodesic Gauss map of spheres and complex projective spaces

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Abstract

For an isometrically immersed submanifold $f : M \to N$, the spherical Gauss map is the induced immersion of the unit normal bundle $UM^{\perp}$ into the unit tangent bundle UN. Compact rank one symmetric spaces have the distinguishing feature that their geodesics are closed with the same period, and so we can define the manifold of geodesics $\mathcal{Q}$ as the quotient of the unit tangent bundle by geodesic flow. Through this quotient we define the geodesic Gauss map $\gamma : UM^{\perp} \to \mathcal{Q}$ to be the Lagrangian immersion given by the projection of the spherical Gauss map. In this thesis we establish relationships between the minimality of isometrically immersed submanifolds of the sphere and complex projective space and the minimality of the geodesic Gauss map with respect to the Kähler-Einstein metric on $\mathcal{Q}$. In particular, we establish that for an isometrically immersed holomorphic submanifold of $\mathbb{CP}^n$, its geodesic Gauss map is minimal Lagrangian if it has conformal shape form.

Contents

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Declaration

This thesis has not previously been accepted in substance for any degree and is not being concurrently submitted in candidature for any degree other than Doctor of Philosophy at the University of York. This thesis is the result of my own investigations, except where otherwise stated. In particular, the contents of Chapter 4 are based on the submitted paper *Minimal Lagrangian submanifolds via the geodesic Gauss map* which contains joint work with Dr Ian McIntosh [8]. Other sources are acknowledged by explicit references.

Introduction

In this thesis we shall study minimal submanifolds of the spheres and complex projective spaces and how we can use a form of Gauss map to relate them to minimal Lagrangian submanifolds. For an isometric immersion, the mean curvature is equal to the tension field in the sense of harmonic map theory. To this end we shall be investigating harmonic isometrically immersed submanifolds.

The Ruh-Vilms theorem [25] is a result which links the parallel mean curvature vector field condition of an isometrically immersed submanifold $f : M \to \mathbb{R}^n$ with the harmonicity of its Gauss map. In the specific case of Euclidean space, we can define the Gauss map to take values in the Grassmannian $\mathrm{Gr}(m, n)$ of all m-dimensional linear subspaces of $\mathbb{R}^n$. The Gauss map associates to each $x \in M$ the m-dimensional subspace $df(T_x M) \in \mathrm{Gr}(m, n)$. The theorem then takes the following form:

Theorem 0.0.1 (Ruh-Vilms)**.** *For an isometrically immersed submanifold $f : M \to \mathbb{R}^n$ with Gauss map $\hat{\gamma}_f$:*

$$\tau(\hat{\gamma}_f) = \nabla H_f,$$

where H_f is the mean curvature normal vector field.

The immediate implication of this theorem is that $\hat{\gamma}_f : M \to \mathrm{Gr}(m, n)$ is a harmonic map if and only if the immersion f has parallel mean curvature. While this definition of the Gauss map only exists for Euclidean space, Obata [21] gave one way of generalising it to spaces of constant curvature. Wood [28] then further extended this to any ambient space as the *Gauss section.* By considering only vertical variations he found a Ruh-Vilms like result, that under certain conditions of the codomain an isometrically immersed submanifold has parallel mean curvature if and only if the Gauss section is a critical point of the vertical energy (a harmonic section).

Another generalisation of the Gauss map to any ambient space was defined by Jensen and Rigoli [14], the spherical Gauss map. The previous Gauss maps assign to each point on the submanifold an analogue of its tangent space. The spherical Gauss map instead considers the immersion of each point of the unit normal bundle into the ambient unit tangent space. Jensen and Rigoli studied harmonicity conditions for the spherical Gauss map with respect to the Sasaki metrics constructed on each of the unit normal bundle and the ambient tangent bundle (in which case the spherical Gauss map is not an isometric immersion). Later, Cintract and Morvan [7] instead considered the entire normal bundle as a submanifold of the ambient tangent bundle. They equipped the ambient tangent bundle with the Sasaki metric and the normal bundle with the pullback of this metric, rather than its own Sasaki metric.

In this thesis we shall consider an alternative approach, the geodesic Gauss map. A compact rank one symmetric space, N, (a CROSS) such as the sphere and complex projective space, has the distinguishing feature that all of its geodesics are closed and of equal length. We can thus construct a quotient space of the unit tangent bundle by the geodesic flow which is itself a manifold, called the manifold of geodesics, $\mathcal{Q}$. We then define the projection of the spherical Gauss map under this quotient to be the geodesic Gauss map, γ. We shall be studying the harmonicity of this map with respect to the pullback to the unit normal bundle of various metrics on UN, which project to $\mathcal{Q}$ such that γ is an isometric immersion.

The main motivation for this map becomes apparent when we use symplectic redution to equip $\mathcal{Q}$ with a symplectic form $\lambda_{\mathcal{Q}}$ induced from the canonical symplectic form on TN. In this case, given an isometrically immersed submanifold $f : M \to N$, the image $\gamma_f(UM^{\perp})$ is a Lagrangian submanifold with respect to the Kähler-Einstein structure on $\mathcal{Q}$. Harmonicity of the geodesic Gauss map with respect to the Kähler-Einstein metric $\lambda_{\mathcal{Q}}(\cdot, J(\cdot))$ would thus allow for construction of minimal Lagrangian submanifolds of $\mathcal{Q}$ from the more adundant examples of minimal submanifolds of compact rank one symmetric spaces.

In Chapter 1 we shall establish a homogeneous geometry approach to working with symmetric spaces. This shall be advantageous, as for a compact rank one symmetric space both its unit tangent bundle and manifold of geodesics are themselves homogeneous spaces. We shall assume that the reader is familiar with both Riemannian geometry (as discussed for example in [16]) and Lie groups (as discussed in [27]). From this foundation we shall establish isomorphisms β and $\hat{\beta}$ between the tangent bundle of a homogeneous space and subbundles of the bundle of Lie algebras arising from the transitive group action over the homogeneous space. This will then allow for easy comparisons between the tangent spaces of the various manifolds. We shall then use these isomorphisms to study the Levi-Civita connection of the naturally reductive metric induced on a homogeneous space by its group action, called the normal metric. For a CROSS, this is the standard symmetric space metric.

In Chapter 2 we shall introduce some concepts from symplectic geometry. In particular we shall define symplectic and contact forms, Kähler structures and symplectic reduction, as these shall be necessary for the construction of the manifold of geodesics. We identify the unit tangent bundle UN of a CROSS with a homogeneous space and describe its isotropy subgroup. We use the canonical symplectic structure on the tangent bundle to define a contact structure for UN for which the Reeb vector field is the geodesic flow vector field. By using symplectic reduction we then construct the manifold of geodesics and introduce its canonical symplectic and complex structures.

We then discuss the Kähler-Einstein structure on the manifold of geodesics. When our ambient manifold is a sphere, the normal metric on $\mathcal{Q} \cong \mathrm{Gr}(2, n+1)$ is compatible with the Kähler-Einstein structure. For the other CROSSes this isn't the case, and so we describe the Kähler-Einstein metric on $\mathcal{Q}$ and define a $\pi_{\mathcal{Q}}$-related metric on the unit normal bundle, which we denote by $h_{\mathcal{Q}}$. We also introduce the Sasaki metric on TN and show that it agrees with the normal metric if and only if N is a sphere.

In Chapter 3, we introduce the concepts of harmonic maps and minimal submanifolds. We then introduce the Ruh-Vilms theorem and discuss previous extensions of the Gauss map and Ruh-Vilms theorem to non-Euclidean ambient spaces. We define the spherical Gauss map μ, and with the manifold of geodesics constructed we are able to use it to define the geodesic Gauss map γ. We then establish that the spherical Gauss map makes $\mu : UM^{\perp} \to UN$ a Legendrian submanifold with respect to the contact distribution of UN. Since the geodesic Gauss map is the projection of the spherical Gauss map to $\mathcal{Q}$, we are then able to prove that γ is harmonic if and only if μ is harmonic.

In Chapter 4, we specialise to the case of a submanifold of a sphere $f : M \to S^n$. This chapter is based on joint work with my supervisor Ian McIntosh [8]. For S^n, the normal metric, Sasaki metric and $h_{\mathcal{Q}}$ are isometric. We calculate the mean curvature of the geodesic Gauss map using a choice of adapted local frames, in order to relate it to the mean curvature of f. However, to obtain a direct relationship requires $\pi^{\perp} : UM^{\perp} \to M$ to be horizontally conformal. The condition of horizontal conformality is equivalent to a condition on the shape operator of f, which in [8] we named *conformal shape form* (referred to in [10] as conformal second fundamental form). This condition amounts to the requirement that the squares of all the eigenvalues of the shape operator are equal. We thus prove the following Ruh-Vilms type result for submanifolds of spheres.

Theorem 0.0.2. *Let $f : M \to (S^n, g)$ be an isometric immersion with respect to the round metric g, and let h_s be the restriction of the Sasaki metric to the unit tangent bundle US^n. Let $\mu : UM^{\perp} \to US^n$ be the spherical Gauss map, and let $\tau(\mu)$ be the tension field of μ with respect to $\mu^* h_s$. For any $\xi \in UM^{\perp}$, let $Z, W \in T_{\xi}(UM^{\perp})$ such that Z is horizontal with respect to $\pi^{\perp} : UM^{\perp} \to M$ and*

W is vertical. If f has conformal shape form, then

$$h_s\left(\tau(\mu), JZ\right) = -\frac{1}{1+r(\xi)^2} g\left(\nabla^{\perp}_{d\pi_N(Z)} H_f, \xi\right),$$
$$h_s\left(\tau(\mu), JW\right) = \frac{1}{1+r(\xi)^2} g\left(H_f, d\pi_N(JW)\right),$$

where $r(\xi)^2 = \frac{1}{dim(M)}\mathrm{tr}A_f(\xi)^2$.

Since μ is an isometric Legendrian immersion and $\tau(\mu)$ is orthogonal to the Reeb vector field this characterises $\tau(\mu)$, and thus we establish that for any minimal surface in a sphere, its geodesic Gauss map is a minimal Lagrangian submanifold of $\mathrm{Gr}(2, n+1)$.

In Chapter 5 we proceed to adapt this result to the case of a complex projective space, $\mathbb{CP}^n$. We consider the three metrics we have used for UN, the normal metric, Sasaki metric and $h_{\mathcal{Q}}$ and see that they now disagree. Since the Sasaki metric doesn't descend to a metric on $\mathcal{Q}$, we now only use it to aid in calculations in UN through comparison with the other two metrics. The normal metric no longer projects to a Kähler-Einstein metric but is easier to work with than $h_{\mathcal{Q}}$, so we still consider both it and $h_{\mathcal{Q}}$.

A vital component of the proof of the sphere case is that the fibres of the spherical Gauss map are totally geodesic, which is a property of the Sasaki metric. We thus consider which submanifolds of $\mathbb{CP}^n$ have a similar property. We establish in the case that the submanifold is either holomorphic or coisotropic, its spherical Gauss map has minimal fibres with respect to both the normal metric and $h_{\mathcal{Q}}$. We then proceed to construct local frames for the immersion $f : M \to \mathbb{CP}^n$ as in the sphere case, taking extra care to make sure the frame of vector fields also respects the standard complex structure $\mathcal{I}$ on $\mathbb{CP}^n$.

In the case of a holomorphic submanifold, the restrictions of the three metrics to $\pi^{\perp}$-horizontal vectors are isometric. We are thus able to adapt the proof of the sphere case with relative ease. Since every holomorphic submanifold is minimal, our next Ruh-Vilms type result takes the following form:

Theorem 0.0.3. *Let $f : M \to \mathbb{CP}^n$ be a holomorphic isometrically immersed submanifold with conformal shape form. Then its spherical and geodesic Gauss maps are minimal for h_Q and minimal Lagrangian for the Kähler-Einstein metric respectively.*

In the case of coisotropic submanifolds, the behaviour of the tension field is further removed from that of the sphere as the differences in curvature begin to manifest. In particular, the conditions of conformal shape form and horizontal conformality for h_Q are mutually exclusive and an additional curvature term appears in the Ruh-Vilms type theorem.

Theorem 0.0.4. *Let $f : M \to (\mathbb{CP}^n, g)$ be an isometrically immersed coisotropic submanifold with respect to the metric h_Q on $U\mathbb{CP}^n$. Let $Z, W \in T_\xi(UM^\perp)$ such that Z is horizontal with respect to $\pi^\perp : UM^\perp \to M$ and W is vertical. If $\pi^\perp$ is horizontally conformal with conformal factor a^2, then*

$$\begin{aligned} h_Q(\tau_Q(\mu), JZ)|_\xi &= -a^2 \left(g\left(\nabla^\perp_{d\pi_N(Z)} H_f, \xi\right) - tr_{f^*g} m((\mathcal{L}^{-1}Z)\right), \\ h_Q(\tau_Q(\mu), JW)|_\xi &= -a^2 \left(g(H_f, d\pi_N(JW)) + tr_{f^*g} m(W)\right), \end{aligned}$$

where

$$\mathcal{L} : TM \to TM; \; X \to \left(1 - \frac{1}{2} g(Z, \mathcal{I}\xi)\right) X,$$

and

$$m(Z) : TM \times TM \to \mathbb{R}; \; X, Y \mapsto g\left(R^N(\xi, A_f(\xi)\mathcal{L}X)Y, d\pi_N(JZ)\right).$$

1 The homogeneous geometry of symmetric spaces

1.1 Reductive homogeneous spaces

In this thesis we shall assume an awareness of the basic properties of differentiable manifolds and Lie groups, as can be found in [27]. In particular, we shall be working with two special types of manifold, homogeneous spaces and symmetric spaces, which are defined with respect to the action of a Lie group. Throughout this thesis, unless otherwise stated, all manifolds are assumed to be smooth and oriented and all maps are smooth.

Definition 1.1.1. *Let G be a compact connected Lie group, and let K be a closed (Lie) subgroup. The coset space*

$$G/K = \{[gK] = \{gk : k \in K\} : g \in G\}$$

is a differentiable manifold. We equip it with a projection

$$\pi_K : G \to G/K;\ g \mapsto [gK].$$

Any manifold of this form is called a homogeneous space.

A particularly common example of a homogeneous manifold that we will make frequent use of is given by the following result from [27].

Theorem 1.1.2. *Let G be a Lie group which acts transitively on the left of a manifold M. Let K be the* isotropy group *of a basepoint $o \in M$ (i.e. $K \subseteq G$ is the set of elements that preserve o under the group action). Then M is diffeomorphic to the homogeneous space G/K.*

A simple example of such a homogeneous space is the n-sphere, S^n. Using the standard embedding in $\mathbb{R}^{n+1}$, we can choose the basepoint to be $o = e_{n+1} = (0, ..., 0, 1)^T$. The group $SO(n+1)$ acts transitively on S^n by rotations and the isotropy subgroup K for e_{n+1} is isomorphic to $SO(n)$. Thus, $S^n \cong SO(n+1)/SO(n)$.

Since K is a Lie subgroup, it has a corresponding Lie subalgebra $\mathfrak{k} \subset \mathfrak{g}$. We will equip G with a *bi-invariant* metric, $\langle \cdot, \cdot \rangle$. This is a metric which is invariant under both the left and right actions of G (and hence also invariant under the adjoint representation, $\mathrm{Ad}_g := dL_g dR_g^{-1}$). The tangent space $T_{[K]}G/K$ can then be identified with $\mathfrak{m} := \mathfrak{k}^{\perp}$.

Definition 1.1.3. *Let G/K be a homogeneous space. We say that G/K is* reductive *if the decomposition*

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$$

is Ad_K-invariant. Equivalently, $[\mathfrak{k}, \mathfrak{m}] \subseteq \mathfrak{m}$.

If the decomposition has the additional property $[\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{k}$, then G/K is globally symmetric.

Note that since K is a Lie subgroup, we have the additional relationship $[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}$.

To illustrate why we use the term symmetric, let us assume that N is a Riemannian manifold (with connection). Such a manifold is *Riemannian symmetric* if every point $p \in N$ is an isolated fixed point of an involutive isometry $\sigma_p : N \to N$. Locally this isometry is given by the map $\sigma_p(\gamma(t)) = \gamma(-t)$, where γ is any geodesic through p and t is sufficiently small. If N is Riemannian symmetric, then it is a homogeneous space of the type given in Theorem 1.1.2, where G is the identity component of the isometry group $I(N)$ and K is the isotropy group of any point. This homogeneous space $N \cong G/K$ is globally symmetric. Conversely, given a globally symmetric space G/K (K compact), there exists a metric such that G/K is Riemannian symmetric. In general, we will simply use *symmetric* to refer to a Riemmanian (and thus globally) symmetric space.

Returning to reductive homogeneous spaces, while we've identified the tangent space at a basepoint to a Lie subalgebra, when it comes to differentiation this is only useful if we take the Lie group approach of restricting ourselves to left-invariant vector fields. In order to work with more general vector fields, we'd like to take a Lie algebra approach which works anywhere on the manifold. In order to do this we shall construct an isomorphism between TN (where $N = G/K$) and a subbundle of the trivial bundle $N \times \mathfrak{g}$. Given a K-invariant subspace $\mathfrak{v} \in \mathfrak{g}$, we define the subbundle

$$[\mathfrak{v}]_K = \{([gK], \mathrm{Ad}_g(\xi)) : g \in G, \xi \in \mathfrak{v}\} \subseteq N \times \mathfrak{g}.$$

We then define the map

$$\beta_K : TN \to [\mathfrak{m}]_K;\ \left.\frac{d}{dt}\right|_{t=0} e^{t\xi}[gK] \mapsto ([gK], \xi).$$

To see that this is an isomorphism, we use the following result [6, Proposition 3.6].

Lemma 1.1.4. *For a Lie group homomorphism $\phi : G_1 \to G_2$, $\phi(e^{\xi}) = e^{d\phi(\xi)}$.*

If we consider the map

$$b : N \times \mathfrak{g} \to TN;\ ([gK], \xi) \mapsto \left.\frac{d}{dt}\right|_{t=0} e^{t\xi}[gK],$$

we observe that the kernel consists of precisely those vectors for which $e^{t\xi}[gK]$ is constant. Since $e^{t\xi}|_{t=0} = e$ (the identity element) this means we require

$$\left.\frac{d}{dt}\right|_{t=0} (g^{-1}e^{t\xi}g) = \mathrm{Ad}_g^{-1}(\xi) \in \mathfrak{k}.$$

Given a vector $\mathrm{Ad}_g(\xi) \in [\mathfrak{k}]_K$, $e^{t\xi}$ is a curve in K, and so

$$\left.\frac{d}{dt}\right|_{t=0} e^{t\mathrm{Ad}_g(\xi)}[gK] = \left.\frac{d}{dt}\right|_{t=0} ge^{t\xi}[K] = \left.\frac{d}{dt}\right|_{t=0} [gK] = 0.$$

Hence, $[\mathfrak{k}]_K$ is the kernel of b and the restriction to $[\mathfrak{m}]_K$ is an isomorphism. Since $\beta_K(b|_{[\mathfrak{m}]_K}) = \mathrm{id}$, β_K is also an isomorphism.

An alternative approach is to identify TN with the associated bundle $G \times_K \mathfrak{m}$ defined by taking the quotient of $G \times \mathfrak{m}$ by the right adjoint action of K on $\mathfrak{m}$,

$$G \times_K \mathfrak{m} = \left\{[g, \xi] = [gk, \mathrm{Ad}_k^{-1}(\xi)]; (g, \xi) \in G \times \mathfrak{m},\ k \in K\right\}.$$

We define a map

$$\hat{\beta}_K : TN \to G \times_K \mathfrak{m}; \ \left.\frac{d}{dt}\right|_{t=0} ge^{t\xi}[K] \mapsto [g, \xi].$$

Again, by considering the map

$$\hat{b} : G \times_K \mathfrak{g}; \ [g, \xi] \mapsto \left.\frac{d}{dt}\right|_{t=0} ge^{t\xi}[K],$$

the kernel is given by $G \times_K \mathfrak{k}$, and so $\hat{\beta}_K = (b|_{G\times_K \mathfrak{m}})^{-1}$ is an isomorphism. By noting that $ge^{t\xi}[K] = e^{t\mathrm{Ad}_g \xi}[gK]$ we can then identify the two approaches such that $\hat{\beta}_K(X) = [g, \xi]$ when $\beta_K(X) = ([gK], \mathrm{Ad}_g \xi)$.

1.2 The Levi-Civita connection of the normal metric

With these isomorphisms in hand we are now in a position to consider the Riemannian geometry of a reductive homogeneous space.

Definition 1.2.1. *Let $N = G/K$ (K compact) be a reductive homogeneous space. Let G be equipped with a bi-invariant metric $\langle \cdot, \cdot \rangle$. At a point $[gK] \in N$, the* normal metric *on N is given by*

$$g(\cdot, \cdot) = \langle \beta_K(\cdot), \beta_K(\cdot) \rangle .$$

By considering vectors in $\mathfrak{m}$ corresponding to vectors in T_pN, we can observe a particularly useful property of the normal metric.

Lemma 1.2.2. *For $\xi, \eta, \zeta \in \mathfrak{m}$,*

$$\langle [\xi, \eta]_{\mathfrak{m}}, \zeta \rangle + \langle \xi, [\zeta, \eta]_{\mathfrak{m}} \rangle = 0, \tag{1}$$

where $V_{\mathfrak{m}}$ denotes the orthogonal projection of V onto $\mathfrak{m} \subset \mathfrak{g}$.

Proof. The adjoint endomorphism ad is defined by $d\mathrm{Ad}$. As shown in [6, Proposition 3.7], $\mathrm{ad}_X(Y) = [X, Y]$. By differentiating the Ad-invariance of the metric, we obtain

$$\langle \mathrm{ad}_\eta(\xi), \zeta \rangle + \langle \xi, \mathrm{ad}_\eta(\zeta) \rangle = 0.$$

□

A homogeneous space equipped with a metric that obeys (1) is said to be *naturally reductive.* An important property of naturally reductive homogeneous spaces is that they have homogeneous geodesics, meaning the geodesic generated by a vector $X \in T_{\pi(g)}N$ takes the form

$$\gamma_X(t) = e^{t\beta_K(X)}[gK]. \tag{2}$$

In order to derive the Levi-Civita connection for the normal metric, we will want to consider the horizontal lift of vector fields on TN to TG. The projection $\pi_K : G \to N = G/K$ gives rise to a decomposition $TG = \mathcal{H}_G \oplus \mathcal{V}_G$, where the vertical bundle $\mathcal{V}_G = \ker(d\pi_N)$ and the horizontal bundle is the orthogonal complement with respect to our bi-invariant metric on G.

Definition 1.2.3. *Given a submersion $\pi : (M, g) \to (N, h)$, we say that π is* Riemannian *if for all horizontal vectors $X^{\mathcal{H}}, Y^{\mathcal{H}}$,*

$$g(X^{\mathcal{H}}, Y^{\mathcal{H}}) = h(d\pi(X^{\mathcal{H}}), d\pi(Y^{\mathcal{H}})).$$

Definition 1.2.4. *A vector field $V \in \Gamma(TG)$ is* basic *with respect to the Riemannian submersion π_N if*

1. *$V \in \Gamma(\mathcal{H}) \subset \Gamma(TG)$,*
2. *V is π_N-related to a vector field $X \in \Gamma(TN)$.*

There is a bijective relationship between basic vector fields in $\Gamma(\mathcal{H})$ and vector fields in $\Gamma(TN)$. Given a vector field $X \in \Gamma(TN)$, the corresponding basic vector field, $\bar{X}$, is the horizontal lift *of X.*

By using the Maurer-Cartan form $\omega = dL_g^{-1}$ to identify each $\mathcal{H}_G|_g$ with $\mathfrak{m}$, we see that by definition the normal metric is Riemannian.

Lemma 1.2.5. *A horizontal vector field $H \in \Gamma(\mathcal{H}_G)$ is basic if and only if $\omega(H)$ is Ad_K-equivariant, i.e.*

$$\omega_{gk}(H_{gk}) = \mathrm{Ad}_k^{-1}\omega_g(H_g)$$

Proof. To prove this result, we will want to consider the following commutative diagram:

$$\begin{tikzcd}
\mathcal{H}_G \arrow[r, "\omega|_{\mathcal{H}_G}"'] \arrow[d, "d\pi_N"'] & G \times \mathfrak{m} \arrow[d, "\tilde{\pi}"] \\
TN \arrow[r, "\hat{\beta}_K"'] & G \times_K \mathfrak{m}
\end{tikzcd}$$

where $\tilde{\pi}$ is the quotient map $(g, \xi) \mapsto [g, \xi]$. Let us assume that $\bar{X}$ is the horizontal lift of a vector field $X \in TN$, and thus basic. Since $d\pi_N(\bar{X}) = X$, we see that $\tilde{\pi}(\omega(\bar{X})) = \hat{\beta}_K(X)$. Hence, for all gk, $\omega_{gk}(\bar{X}_{gk}) \in [g, X_g]$.

Conversely, assume for a horizontal vector field H, $\omega(H)$ is Ad_K-equivariant. Then,

$$\begin{aligned}
\omega_{gk}(H_{gk}) &= \mathrm{Ad}_k^{-1}\omega_g(H_g) \\
&= dL_k^{-1} dL_g^{-1} dR_k(H_g) \\
&= \omega_{gk}(dR_k H_g).
\end{aligned}$$

Considering $\hat{\beta}_K \circ d\pi_N$:

$$\left.\frac{d}{dt}\right|_{t=0} gke^{tH_{gk}}[K] = \left.\frac{d}{dt}\right|_{t=0} gke^{tdR_k H_g}[K] = \left.\frac{d}{dt}\right|_{t=0} ge^{t\mathrm{Ad}_k^{-1} H_g}[K],$$

and so $[gk, H_{gk}] = [g, \mathrm{Ad}_k^{-1} H_g]$. Thus $d\pi_N(H_{gk}) = d\pi_N(H_g)$, and H is basic. $\square$

In order to describe the Levi-Civita connection for TN in terms of β_K and $\hat{\beta}_K$, we shall need two more results:

Lemma 1.2.6. *[23, Lemma 1] Given a Riemannian submersion $\pi : M \to N$, if $\bar{X}, \bar{Y} \in \Gamma(TM)$ are basic vector fields over $X, Y \in \Gamma(TN)$ then*

$$d\pi(\nabla^M_{\bar{X}}\bar{Y}) = \nabla^N_X Y.$$

Lemma 1.2.7. *[6, Corollary 3.19] Let X, Y be left-invariant vector fields in $\Gamma(TG)$. Then if $\langle \cdot, \cdot \rangle$ is a left invariant metric and ∇^G is the associated Levi-Civita connection,*

$$\nabla^G_X Y = \frac{1}{2}\left([X,Y] - \mathrm{ad}^*_X Y - \mathrm{ad}^*_Y X\right),$$

where $\mathrm{ad}^*_X Y$ is defined such that for all $Z \in \Gamma(TG)$:

$$\langle \mathrm{ad}^*_X Y, Z\rangle = \langle Y, \mathrm{ad}_X(Z)\rangle .$$

In order to make use of this last lemma, we will need to adjust it to work for any vector field in $\Gamma(TG)$. To do this, we can choose $E_1, ..., E_m$ to be a basis of left-invariant vector fields for G. Then, for $X, Y \in \Gamma(TG)$:

$$\begin{aligned}
\nabla^G_X Y &= \sum_{i,j=1}^{m} X^i \nabla_{E_i}(Y^j E_j) \\
&= \sum_{i,j=1}^{m} X^i (E_i Y^j) E_j + \frac{1}{2} X^i Y^j \left([E_i, E_j] - \mathrm{ad}^*_{E_i} E_j - \mathrm{ad}^*_{E_j} E_i\right).
\end{aligned}$$

Since E_i, E_j are left-invariant vector fields, $\omega(E_j)$ is constant and $\omega([E_i, E_j]) = [\omega(E_i), \omega(E_j)]$, where the bracket on the left is the Lie-bracket of vector fields and the bracket on the right is the Lie algebra bracket associated with the Lie group G. Since this is defined on $\mathfrak{g}$, not $\Gamma(TG)$, the functions X^i, Y^j can now pass through the bracket and thus

$$\omega(\nabla^G_X Y) = X\omega(Y) + \frac{1}{2}\left([\omega(X), \omega(Y)] - \mathrm{ad}^*_{\omega(X)}\omega(Y) - \mathrm{ad}^*_{\omega(Y)}\omega(X)\right). \tag{3}$$

By untwisting $\hat{\beta}_K(X)$ and thinking of it as an Ad_K-equivariant map from $G \to \mathfrak{m}$ such that $\hat{\beta}_K(X)_{gk} = \mathrm{Ad}_k^{-1}\hat{\beta}_K(X)_g$, then we can view $\pi_{\mathfrak{m}}\omega_g(\bar{X}_g)$ as $\hat{\beta}_K(X)|_g$ (where $\pi_{\mathfrak{m}}$ is the natural projection from $G \times \mathfrak{g} \to G \times \mathfrak{m}$). Let g denote a left-invariant metric on G/K for which $\pi_N : G \to N = G/K$ is a Riemannian submersion, and let ∇^N denote its Levi-Civita connection. Then, using Lemma 1.2.6, for basic vector fields $\bar{X}, \bar{Y} \in \Gamma(TG)$, since $[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}$, (3) implies

$$\begin{aligned}\hat{\beta}_K(\nabla_X Y) = \bar{X}\hat{\beta}_K(Y) + \frac{1}{2}P_{\mathfrak{m}}\Big([\hat{\beta}_K(X), \hat{\beta}_K(Y)] - \mathrm{ad}^*_{\hat{\beta}_K(X)}\hat{\beta}_K(Y)\\ -\mathrm{ad}^*_{\hat{\beta}_K(Y)}\hat{\beta}_K(X)\Big),\end{aligned} \tag{4}$$

where $P_{\mathfrak{m}}$ is the projection onto $[\mathfrak{m}]_K \subset N \times \mathfrak{g}$.

In this untwisted form, $\hat{\beta}_K(X)_g = \mathrm{Ad}_g^{-1}\beta_K(X)_{gK}$. Since $d\pi_N(\bar{X}_g) = X_{gK}$, the curve $ge^{t\hat{\beta}_K(X)_g} = e^{t\beta_K(X)_{gK}}g \subset G$ has tangent vector $\bar{X}_g$ at $t = 0$. Hence the first term in (4) becomes

$$\begin{aligned}\bar{X}_g\left(\mathrm{Ad}_g^{-1}\beta_K(X)_{gK}\right)_{\mathfrak{m}} &= \frac{d}{dt}\Big|_{t=0}\left(\mathrm{Ad}^{-1}(e^{t\beta_K(X)_{gK}}g)(\beta_K(Y)_{e^{t\beta_K(Y)_{gK}}g})\right)_{\mathfrak{m}}\\ &= \mathrm{Ad}_g^{-1}\left(\left(\frac{d}{dt}\Big|_{t=0}\mathrm{Ad}_{e^{-t\beta_K(X)_{gK}}}\right)\beta_K(Y)_{gK} + X_g\beta_K(Y)_{gK}\right)_{\mathfrak{m}}\\ &= \mathrm{Ad}_g^{-1}\left(-\mathrm{ad}_{\beta_K(X)_{gK}}\beta_K(Y)_{gK} + X\beta_K(Y)_{gK}\right)_{\mathfrak{m}}\end{aligned}$$

Since $\mathrm{Ad}[\cdot,\cdot] = [\mathrm{Ad}(\cdot), \mathrm{Ad}(\cdot)]$, we have proved the following result.

Lemma 1.2.8. *Given a left-invariant metric $\langle\cdot,\cdot\rangle$ on G, let g denote a G-invariant metric on $N = G/K$ which makes $\pi_N : G \to G/K$ a Riemannian submersion. The Levi-Civita connection for g takes the form*

$$\beta_K(\nabla^N_X Y) = P_{\mathfrak{m}}\left(X\beta_K(Y) - \frac{1}{2}\left([\beta_K(X), \beta_K(Y)] + \mathrm{ad}^*_{\beta(X)}\beta(Y) + \mathrm{ad}^*_{\beta(Y)}\beta(X)\right)\right), \tag{5}$$

where $P_{\mathfrak{m}} : N \times \mathfrak{g} \to [\mathfrak{m}]_K \subseteq N \times \mathfrak{g}$ is the orthogonal projection.

Corollary 1.2.9. *When g is the normal metric induced by a bi-invariant metric $\langle\cdot,\cdot\rangle$,*

$$\beta_K(\nabla^N_X Y) = P_{\mathfrak{m}}\left(X\beta_K(Y) - \frac{1}{2}[\beta_K(X), \beta_K(Y)]\right)$$

Proof. By (1), $\mathrm{ad}^*_{\beta(X)}\beta(Y) + \mathrm{ad}^*_{\beta(Y)}\beta(X) = 0$. $\square$

For the most part our interest in N is going to be focused on isometrically immersed submanifolds $f : M \to N$. It is therefore important we understand the corresponding pullback connection along f. In order to do this, we shall want to work with a local frame $F : U \to G$ on an open contractible neighbourhood $U \subseteq M$ such that $f = \pi_K \circ F$. We shall make much use of the Lie algebra-valued 1-form $\alpha = F^*\omega : TU \to \mathfrak{g}$. By considering the untwisted form of $\hat{\beta}_K$, we observe

$$\begin{aligned} f^*\beta_K = \beta_K(d(\pi_N \circ F)) &= \mathrm{Ad}_F((\hat{\beta}_K \circ d\pi_N)(dF)) \\ &= \mathrm{Ad}_F(\pi_{\mathfrak{m}}\omega(dF)) = \mathrm{Ad}_F\alpha_{\mathfrak{m}}. \end{aligned} \tag{6}$$

For clarity of notation, we shall henceforth define $\alpha_{\mathfrak{m}}(X) = \alpha(X)_{\mathfrak{m}}$.

Lemma 1.2.10. *Let $f : M \to N = G/K$ be an isometrically immersed submanifold, where G is a matrix Lie group and N is equipped with a left-invariant metric such that $\pi : G \to G/K$ is a Riemannian submersion. Then for $X \in \Gamma(TM),\ V \in \Gamma(M \times \mathfrak{g})$,*

$$X\mathrm{Ad}_F(V) = \mathrm{Ad}_F((XV) + ([\alpha(X), V])), \tag{7}$$

where $F : U \to G$ is a local frame compatible with f and $\alpha = F^*\omega$.

Proof. The first term comes directly from the chain rule. To acquire the second term, we want to consider $[\alpha(X), V]$. Since we have assumed G to be a matrix Lie group, the pullback of the Maurer-Cartan form is given by $\alpha = F^{-1}dF$, where we are considering "F" as the action of $g(F)$ on elements of $\mathfrak{g}$ by matrix multiplication. For a matrix Lie group, $dL_g(X) = gX$, and so $\mathrm{Ad}_F(X) = FXF^{-1}$. Thus,

$$\begin{aligned} \mathrm{Ad}_F[\alpha(X), V] = \mathrm{Ad}_F(d\mathrm{Ad}_{\alpha(X)}(V)) &= \mathrm{Ad}_F\left(\left.\frac{d}{dt}\right|_{t=0} e^{t\alpha(X)}Ve^{-t\alpha(X)}\right) \\ &= \mathrm{Ad}_F\left(\left.\frac{d}{dt}\right|_{t=0} \sum_{k,l=0}^{\infty}\left(\frac{t^k\alpha(X)^k}{k!}V\frac{(-t)^l\alpha(X)^l}{l!}\right)\right) \\ &= \mathrm{Ad}_F(\alpha(X)V - V\alpha(X)) \\ &= dF(X)VF^{-1} - FVFdF(X)F^{-1} \end{aligned}$$

Since $(dF)F^{-1} + Fd(F^{-1}) = d(FF^{-1}) = 0$,

$$\begin{aligned}\mathrm{Ad}_F([\alpha(X), V]) &= dF(X)VF^{-1} - FVd(F^{-1})(X) \\ &= \left.\frac{d}{dt}\right|_{t=0} \left(F_{\exp(tX)}VF^{-1}_{\exp(tX)}\right) \\ &= (X\mathrm{Ad}_F)(V)\end{aligned}$$

□

Corollary 1.2.11. *The f-pullback of the Levi-Civita connection takes the form*

$$\begin{aligned}\beta_K(\nabla^f_X df(Y)) = \mathrm{Ad}_F \Bigg(& X\alpha_{\mathfrak{m}}(Y) + [\alpha_{\mathfrak{k}}(X), \alpha_{\mathfrak{m}}(Y)] + \frac{1}{2}[\alpha_{\mathfrak{m}}(X), \alpha_{\mathfrak{m}}(Y)]_{\mathfrak{m}} \\ & -\frac{1}{2}\left(\mathrm{ad}^*_{\alpha_{\mathfrak{m}}(X)}\alpha_{\mathfrak{m}}(Y) + \mathrm{ad}^*_{\alpha_{\mathfrak{m}}(Y)}\alpha_{\mathfrak{m}}(X)\right)_{\mathfrak{m}}\Bigg).\end{aligned}$$

For the normal metric this simplifies to

$$\beta_K(\nabla^f_X df(Y)) = \mathrm{Ad}_F \left(X\alpha_{\mathfrak{m}}(Y) + [\alpha_{\mathfrak{k}}(X), \alpha_{\mathfrak{m}}(Y)] + \frac{1}{2}[\alpha_{\mathfrak{m}}(X), \alpha_{\mathfrak{m}}(Y)]_{\mathfrak{m}} \right)$$

Proof. Using (5) and (7), we acquire

$$\beta_K(\nabla^f_X df(Y)) = \mathrm{Ad}_F \left(X\alpha_{\mathfrak{m}}(Y) + [\alpha(X), \alpha_{\mathfrak{m}}(Y)]_{\mathfrak{m}} - \frac{1}{2}[\alpha_{\mathfrak{m}}(X), \alpha_{\mathfrak{m}}(Y)]_{\mathfrak{m}} \right).$$

Since $[\mathfrak{k}, \mathfrak{m}] \subseteq \mathfrak{m}$ for reductive spaces, we can split the second term into $[\alpha_{\mathfrak{k}}(X), \alpha_{\mathfrak{m}}(Y)] + [\alpha_{\mathfrak{m}}(X), \alpha_{\mathfrak{m}}(Y)]_{\mathfrak{m}}$. □

1.3 Symmetric spaces

Given a compact symmetric space G/K, as shown in [6, Proposition 3.39], the Killing form B provides a bi-invariant metric

$$-B(X, Y) = -\mathrm{tr}(\mathrm{ad}_X \mathrm{ad}_Y)$$

on G. For a compact matrix Lie group, the Killing form is proportional to $\mathrm{tr}(XY)$, and so we can choose our bi-invariant metric on G to be

$$\langle X, Y\rangle = -\frac{1}{2}\mathrm{tr}(XY)$$

(while this is proportional to B, the factor of a half isn't necessarily the coefficient of proportionality but is our choice for later ease) We shall henceforth call this metric the *Killing form metric.*

As with a more general reductive homogeneous space, we can use the Killing form metric to define a normal metric on G/K. Because a symmetric space has the additional condition that $[\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{k}$, the Levi-Civita connection for this metric reduces to $\beta_K(\nabla^N_X Y) = P_{\mathfrak{m}}(X\beta_K(Y))$.

We are able to calculate the curvature by using a second metric connection, defined on any reductive homogeneous space, called the *canonical connection.* This given by $\beta_K(\nabla^{\text{can}}_X Y) = P_{\mathfrak{m}} X \beta_K(Y)$. While the canonical connection is normally only metric and not torsion free, as we can see in the case of a symmetric space it coincides with the Levi-Civita connection. Hence we can find the curvature of a symmetric space from the following result.

Lemma 1.3.1. *[5, Corollary 1.4] The canonical connection on a reductive homogeneous space has the following torsion and curvature:*

$$\beta_K(T(X,Y)) = -P_{\mathfrak{m}}[\beta_K(X), \beta_K(Y)]$$
$$\beta_K(R(X,Y)Z) = -[P_{\mathfrak{k}}[\beta_K(X), \beta_K(Y)], \beta_K(Z)]$$

For a symmetric space we can thus observe that the torsion vanishes, and $\beta_K(R(X,Y)Z) = -[[\beta_K(X), \beta_K(Y)], \beta_K(Z)]$.

Remark 1.3.2. Here and throughout we have used the curvature convention

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Example 1.3.3. Let us consider the sphere S^n. The special orthogonal group $SO(n+1)$ acts transitively on S^n, and $K \cong SO(n)$ is the isotropy group of the basepoint e_{n+1} (where $S^n \subset \mathbb{R}^{n+1} = Span\{e_1, ..., e_{n+1}\}$), so $S^n \cong SO(n+1)/K$. Since $\mathfrak{so}(n)$ is the space of all skew-symmetric real $n \times n$ matrices, we can observe that $\mathfrak{m}$ consists of matrices of the form

$$\underline{X} = \begin{pmatrix} 0_{n\times n} & X \\ -X^t & 0 \end{pmatrix}, \; X \in \mathbb{R}^n.$$

We can easily verify that $[\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{k}$ and thus S^n is symmetric. The normal metric is of the form $\langle \underline{X}, \underline{Y} \rangle = -\frac{1}{2} tr(\underline{XY}) = -\frac{1}{2}(tr(-XY^t) - X^tY) = X \cdot Y$. Thus by identifying $\mathfrak{m}$ with $T_{e_{n+1}} S^n \in T_{e_{n+1}} \mathbb{R}^{n+1}$ we can observe that it is simply the round metric.

2 The Kähler-Einstein structure on the manifold of geodesics

2.1 Symplectic and complex structures

In order to work with the manifold of geodesics, we are going to want to consider some additional structures on the tangent bundle. First, symplectic structures.

Definition 2.1.1. *Let $\omega \in \Omega^2(N)$ be a smooth 2-form over a manifold M. The pair (N, ω) is a* symplectic manifold *if ω obeys the following conditions:*

1. *$d\omega = 0$ (ω is closed),*

2. *if $\omega(v, \cdot) = 0$, then $v = 0$ (ω is non-degenerate).*

We call ω a symplectic structure *on N, and say that each (T_pN, ω_p) is a* symplectic vector space*.*

A consequence of the combination of skew-symmetry and non-degeneracy is that N must be even dimensional.

Definition 2.1.2. *Given symplectic manifolds $(M, \omega), (N, \omega')$, a diffeomorphism $\psi : M \to N$ is a* symplectomorphism *if $\omega = \psi^* \omega'$*

Given a subspace $U \subseteq (V, \omega)$ of a symplectic vector space, we define its *symplectic complement*

$$U^\omega = \{v \in V : \omega(U, v) = 0\}.$$

Two useful properties of the symplectic complement are that $\dim(U) + \dim(U^\omega) = \dim(V)$, and $(U^\omega)^\omega = U$ [18, Lemma 2.2].

Definition 2.1.3. *Let $M \subseteq (N, \omega)$ be a submanifold of a symplectic manifold. Then M is:*

1. isotropic *if* $T_pM \subseteq (T_pM)^\omega$*,*

2. coisotropic *if* $(T_pM)^\omega \subseteq T_pM$*,*

3. Lagrangian *if* $T_pM = (T_pM)^\omega$*,*

4. symplectic *if* $T_pM \cap (T_pM)^\omega = \{0\}$

for all $p \in M$*.*

The restriction of ω to a symplectic submanifold is non-degenerate and thus makes it a symplectic manifold in its own right. We can also observe that Lagrangian submanifolds are the maximal submanifolds on which ω is degenerate, and that they have dimension $\frac{1}{2}\dim(N)$.

A related concept, but for odd dimensional manifolds, is that of a contact structure.

Definition 2.1.4. *Let* N *be a manifold and* $\mathcal{C} \subset TN$ *be a smooth distribution of hyperplanes, locally described as* $\ker\theta$ *for a* 1*-form* $\theta \in \Omega^1(M)$*. If* $d\theta$ *is non-degenerate on* $\mathcal{C}$*, then* $\mathcal{C}$ *is a* contact structure *on* N *(and* θ *is a* contact form *for* $\mathcal{C}$*).*

The non-degeneracy condition on $d\theta$ means that each $(\mathcal{C}_p, d\theta_p)$ is a symplectic vector space, and thus N must be odd dimensional. The choice of a contact form θ for $\mathcal{C}$ defines a vector field $\xi \in \Gamma(TN)$ as the unique vector field such that $d\theta(\xi, \cdot) = 0,\ \theta(\xi) = 1$. This vector field is called the *Reeb vector field.*

Definition 2.1.5. *Let* N *be a* $2n+1$ *dimensional manifold with contact structure* $\mathcal{C}$*. An* n*-dimensional submanifold* $M \subset N$ *is* Legendrian *if* $TM \subset \mathcal{C}$*.*

Lemma 2.1.6. *[18, Proposition 3.42] Let* N *be a* $2n+1$*-dimensional with contact structure* $\mathcal{C}$*. Let* $M \subset N$ *be a submanifold such that* $TM \subset \mathcal{C}$*. At each point* $T_pM \subset \mathcal{C}_p$ *is an isotropic subspace with respect to the symplectic structure* $d\theta$*. In particular, when dim*$(M) = n$*, then* T_pM *is Lagrangian.*

Finally, we define a complex structure.

Definition 2.1.7. *Let N be a $2n$-dimensional manifold. An* almost complex structure *on N is an endomorpism $J \in \Gamma(\mathrm{End}(TN))$ for which $J^2 = -id$. If there exists an atlas (a, U_a) for N such that $da_p(J)$ is the standard complex multiplication given by $\begin{pmatrix} 0_n & -id_n \\ id_n & 0_n \end{pmatrix}$ on $\mathbb{R}^{2n} = \mathbb{C}$ for all a, p, then J is an* (integrable) complex structure *on N.*

In order to avoid having to construct such an atlas, the following equivalent condition for integrability is often used

Theorem 2.1.8. *[18, Theorem 4.12] The* Nijenhuis tensor $N_J : TN \times TN \to TN$ *associated to an almost complex structure J is given by*

$$N_J(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y].$$

The almost complex structure is integrable if and only if $N_J = 0$.

We shall be particularly interested in manifolds for which the metric, symplectic and complex structures interact nicely

Definition 2.1.9. *Let (N, ω) by a symplectic manifold with almost complex structure J. The almost complex structure J is* compatible *with the symplectic structure if:*

1. $\omega(JX, JY) = \omega(X, Y)$,
2. $\omega(X, JX) > 0$

for all $X \neq 0 \in TN$. If J is integrable, then the triple (N, ω, J) is a Kähler structure *on M.*

A Kähler structure induces a compatible metric on M, given by

$$g_J(X, Y) = \omega(X, JY),$$

for which J is an isometry. With this in mind we can reformulate Definition 2.1.3 in terms of the complex structure.

Lemma 2.1.10. *Let $M \subset (N, \omega, J)$ be a submanifold of a Kähler manifold. Then M is:*

1. *isotropic iff $JTM \subseteq T^{\perp}M$,*

2. *coisotropic iff $JTM^{\perp} \subseteq TM$,*

3. *Lagrangian iff $JTM = TM^{\perp}$*

4. *complex iff $JTM = TM$ (in which case we say M is* holomorphic*) .*

Proof. For $X \in TM^{\perp}$, $g_J(TM, X) = 0 = \omega(TM, JX)$. Similarly, for $Y \in TM^{\omega}$, $\omega(TM, Y) = 0 = -g_J(JTM, Y)$. Hence, $TM^{\omega} = JTM^{\perp}$. □

The compatible metric g_J provides us with another interpretation of the integrability of J.

Theorem 2.1.11. *[18, Lemma 4.15] Let $\omega \in \Omega^2(N)$ be a non-degenerate 2-form with a compatible almost complex structure J. Let ∇ denote the Levi-Civita connection for the compatible metric g_J. Then $\nabla J = 0$ if and only if ω is closed and J is integrable.*

Given a symplectic manifold (N, ω), we can use a process called symplectic reduction to construct lower dimensional symplectic manifolds by way of circle actions on TN. In order to do this, we first introduce the concept of Hamiltonian vector fields.

Definition 2.1.12. *Let (N, ω) be a symplectic manifold and $H : N \to \mathbb{R}$ a smooth function. We can define a vector field $X_H \in \Gamma(TN)$ from the identity*

$$\omega(X_H, \cdot) = dH(\cdot).$$

A vector field which can be constructed in this manner is called the Hamiltonian vector field X_H *associated to the* Hamiltonian function H*. If* N *is complete, the flow generated by the vector field* X_H *(the one-parameter group of symplectomorphism* $\varphi : N \times \mathbb{R} \to N$ *such that* $\varphi^0 = \mathrm{id}$ *and* $\frac{d}{dt}\varphi_p^t = X_H|_{\varphi_p^t}$*) is called the* Hamiltonian flow *associated with* H*.*

Hamiltonian vector fields are thus those vector fields for which $\iota_X\omega$ is exact. Since $X_H H = \omega(X_H, X_H) = 0$, the Hamiltonian flow preserves the level sets $H^{-1}(c)$.

In the case that the Hamiltonian flow consists of a periodic family of symplectomorphisms $\psi : M \times \mathbb{R} \to M;\ \psi^1 = \psi^0$, we say that it is a *Hamiltonian action* of S^1 on M.

Lemma 2.1.13. *[18, Lemma 5.2] Suppose that there exists a Hamiltonian action* ψ *of* S^1 *on* (N, ω) *with associated Hamiltionian function* H*. If* S^1 *acts freely on the level set* $H^{-1}(\lambda)$ *then the quotient*

$$\pi_S : H^{-1}(\lambda) \to H^{-1}(\lambda)/S^1$$

is a symplectic manifold with symplectic form ω_H *such that* $\pi_S^*\omega_p = \omega_H|_{\pi_S(p)}$*.*

To see that ω_H is well defined, note that since ψ is a family of symplectomorphisms, $\omega_p = (\psi^t)^*\omega_{\psi_p^t}$ and thus $\omega_H|_{[p]} = \omega_H|_{[\psi_p^t]}$.

2.2 The unit sphere bundle for a CROSS

We shall now restrict our attention to a special class of symmetric spaces, the *compact rank one symmetric spaces* (or *CROSSes*).

Definition 2.2.1. *The* rank *of a symmetric space* G/K *is the dimension of its maximal flat, totally geodesic submanifold.*

This means that the curvature tensor (with respect to the Killing form metric) vanishes over the submanifold and geodesics within the submanifold are also geodesics of the ambient space. Equivalently, the rank of G/K is the dimension of the maximal subspace of $T_o(G/K) \cong \mathfrak{m}$ on which the Lie bracket vanishes. In the case that the rank is 1, the only such submanfiolds are the geodesics themselves. The simple compact rank one symmetric spaces are classified as

$$\begin{aligned} S^n &\cong SO(n+1)/SO(n), \\ \mathbb{RP}^n &\cong SO(n+1)/O(n), \\ \mathbb{CP}^n &\cong SU(n+1)/S(U_n \times U_1), \\ \mathbb{HP}^n &\cong Sp(n+1)/Sp(n) \times Sp(1)), \\ \mathbb{C}a\mathbb{P}^2 &\cong F_4/Spin(9). \end{aligned}$$

Compact rank one symmetric spaces are particularly useful for us to work with as their unit tangent bundles are themselves reductive homogeneous spaces, as we shall prove using the following result.

Lemma 2.2.2. *[13, Ch.X-G] Let N be a Riemannian manifold with isotropy group K for a basepoint o. The isotropy group K acts transitively on the unit sphere U_oN if and only if N is either a Euclidean space or a rank one symmetric space.*

Lemma 2.2.3. *Given a compact rank one symmetric space $N \cong G/K$, the unit tangent bundle UN is a reductive homogeneous space.*

Proof. The Lie group G acts on $TN \cong [\mathfrak{m}] \subseteq G/K \times \mathfrak{g}$ by

$$g \cdot ([hK], \xi) = ([ghK], \mathrm{Ad}_g(\xi)).$$

To show that this action is transitive on $UN \subset TN$, we first fix a basepoint $([K], \nu_o) \in \beta_K(U_oN)$ such that $\nu_0 \in \mathfrak{m}$. The fibre $[\mathfrak{m}]_K|_{[gK]} = \mathrm{Ad}_g\mathfrak{m}$, and so given any $\xi \in U_{[gK]}N$, we observe $\mathrm{Ad}_{g^{-1}}\xi \in U_oN \subset \mathfrak{m}$. By Lemma 2.2.2 we know K acts transitively on U_oN, therefore there exists $k \in K$ such that

$$\mathrm{Ad}_k\nu_0 = \mathrm{Ad}_{g^{-1}}\xi.$$

Hence, for any $([gK], \xi) \in \beta_K(UN)$ there exists $gk \in G$ such that

$$gk \cdot ([K], \nu_o) = ([gkK], \mathrm{Ad}_{gk}\nu_o) = ([gK], \xi). \tag{8}$$

Thus, G acts transitively on UN and so UN is a homogeneous space G/H, where H is the isotropy group of $([K], \nu_o)$. Since for each CROSS G is semisimple, it can be equipped with a the bi-invariant Killing form metric. The decomposition $\mathfrak{h} \oplus \mathfrak{h}^\perp$ is Ad_H-invariant, and thus UN is reductive. $\square$

Because $H \subseteq K$, we can decompose $\mathfrak{g}$ as

$$\begin{aligned}
\mathfrak{g} &= \mathfrak{k} \oplus \mathfrak{m} \\
&= \mathfrak{h} \oplus \mathfrak{h}^\perp =: \mathfrak{h} \oplus \mathfrak{p} \\
&= \mathfrak{h} \oplus (\mathfrak{k} \ominus \mathfrak{h}) \oplus \mathfrak{m} =: \mathfrak{h} \oplus \mathfrak{n} \oplus \mathfrak{m}.
\end{aligned}$$

The following commutative diagram demonstrates the relationship between G/H and G/K.

$$\begin{tikzcd}
 & G/H = UN \arrow[d, "\pi_N|_{UN}"] \\
G \arrow[ur, "\pi_H"] \arrow[dr, "\pi_K"'] & \\
 & G/K = N
\end{tikzcd}
\qquad
\begin{tikzcd}
TUN \arrow[r, "\beta_H"] \arrow[d, "d\pi_N|_{TUN}"'] & {[\mathfrak{p}]_H} \arrow[d] \\
TN \arrow[r, "\beta_K"] & {[\mathfrak{m}]_K}
\end{tikzcd}$$

Since the vertical bundle $\beta_H(\ker(d\pi_N|_{TUN})) = [\mathfrak{n}]_H$, we can define a horizontal bundle for TUN such that $\beta_H(\mathcal{V} \oplus \mathcal{H}) = [\mathfrak{n}]_H \oplus [\mathfrak{m}]_H$. Since UN is reductive and N is symmetric, we obtain the following identities:

1. $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$
2. $[\mathfrak{h}, \mathfrak{n}] \subseteq \mathfrak{p} \cap \mathfrak{k} = \mathfrak{n}$
3. $[\mathfrak{h}, \mathfrak{k}] \subseteq \mathfrak{m}$
4. $[\mathfrak{n}, \mathfrak{n}] \subseteq \mathfrak{k}$
5. $[\mathfrak{n}, \mathfrak{m}] \subseteq \mathfrak{m}$

6. $[\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{k}$.

We shall now examine the canonical contact structure on UN. We begin with the *canonical one-form* $\theta \in \Gamma(T^*(TN))$. If g is a metric for N, then

$$\theta(X) := g(\pi_{TN}(X), d\pi_N(X)). \tag{9}$$

The *canonical symplectic structure* on TN is then given by $\lambda_{TN} := -d\theta$.

If we consider the restriction of θ to UN, its kernel gives rise to a contact structure $\mathcal{C} \in UN$. As shown in [3, 1-G] the Reeb vector field $Z_g \in \Gamma(T(UM))$ such that $d\theta(Z_g, \cdot) = 0$, $\theta(Z_g) = 1$ is given by the *geodesic flow vector field* associated with g. This is the Hamiltonian vector field associated with the energy functional $e(\xi) = g(\xi, \xi)$, such that $d\theta(Z_g, \cdot) = -de(\cdot)$.

To visualise Z_g we can consider the geodesic flow itself. Given a vector $X_p \in T_pN$, it generates a geodesic γ_X in N. The *geodesic flow* is then the one-parameter group of diffeomorphisms $\zeta : \mathbb{R} \times TN \to TN$ such that $\zeta_X^t = \frac{d}{dt}|_{t=0}\gamma_X(t)$. The geodesic flow vector field is then given by $Z_g|_X = \frac{d}{dt}|_{t=0}\zeta_X^t$ (and hence $\theta(Z_g) = g(\gamma'(0), \gamma'(0)) = 1$).

2.3 The manifold of geodesics

Using the geodesic flow, we shall use symplectic reduction to construct a new manifold associated to each compact rank one symmetric space, the *manifold of geodesics*. A geodesic $\gamma \in \mathbb{R} \times N$ is *closed* if there exists a constant $l > 0$ such that for all $t \in \mathbb{R}$, $\gamma(t + l) = \gamma(t)$. If in addition the restriction of γ to $(0, l]$ is not self intersecting, γ is *simply closed* with length l.

Lemma 2.3.1. *[12, Proposition 5.3] Let N be a compact rank one symmetric space. Then all the unit speed geodesics in N are simply closed and have the same length.*

The geodesics generated by unit vectors are unit speed, and so the geodesic flow will preserve UN. The preceeding lemma thus allows us to define a free Hamiltonian action (one such that $g.x = x$ implies $g = \mathrm{id}$) of the circle group S^1 on UN by

$$S^1 \times UN \to UN;\ (e^{it}, U_p) \mapsto \left.\frac{d}{dt}\right|_{t=\theta} \gamma_{U_p}\left(\frac{lt}{2\pi}\right).$$

The unit sphere bundle is precisely the level set $e^{-1}(1)$ of the energy functional and so by Lemma 2.1.13, we can use symplectic reduction.

Definition 2.3.2. *Let N be a compact rank one symmetric space. The* manifold of geodesics *of N is the symplectic manifold*

$$(\mathcal{Q}, \lambda_{\mathcal{Q}}) := (e^{-1}(1)/S^1, \pi_{\mathcal{Q}}^* \lambda_{UN}),$$

where λ_{UN} is the restriction of the canonical symplectic form on the tangent bundle to the unit tangent bundle, and $e(\cdot) = \frac{1}{2}g(\cdot,\cdot)$ is the energy functional on TN.

Since the isotropy group K acts transitively on the fibres of the unit sphere bundle for a CROSS, we can combine (2) and (8) to observe that every geodesic on N takes the form $\gamma(t) = ge^{t\nu_0}[K]$ for some $g \in G$, where $([K], \nu_0)$ is the basepoint for $UN = G/H$. The circle action on UN thus coincides with the action of a circle subgroup on G defined by right multiplication by $S = \{e^{t\nu_0} : t \in \mathbb{R}\}$. In this way we can view the manifold of geodesics as a homogeneous space, $\mathcal{Q} = G/(H \times S) =: G/L$.

This homogeneous space has its own reductive decompostition $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{q}$. If we introduce the additional notation $\mathfrak{s} = \{t\nu_0 : t \in \mathbb{R}\}$ for the Lie algebra of $S \subset G$, we can further decompose $\mathfrak{g}$ as

$$\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{q} = (\mathfrak{h} \oplus \mathfrak{s}) \oplus (\mathfrak{p} \cap \mathfrak{s}^{\perp}).$$

We can achieve a further useful decomposition by considering the operator

ad_{ν_0} on $\mathfrak{g}$. The map

$$K \to \mathfrak{um} = \{X \in \mathfrak{m} : \langle X, X \rangle = 1\}\,;\ k \mapsto \mathrm{Ad}_k(\nu_0)$$

is onto, since Ad_K acts transitively on the unit sphere, and so its differential, $\mathrm{ad}_{\nu_0}|_{\mathfrak{k}}$ is also onto. The tangent space $T_{\nu_0}\mathfrak{um}$ identified with $\mathfrak{m} \cap \mathfrak{s}^{\perp}$ consists of all vectors orthogonal to ν_0. If we define $\mathfrak{m}_0 := \mathfrak{m} \cap \mathfrak{s}^{\perp}$, we thus have $\mathrm{ad}_{\nu_0}(\mathfrak{k}) = \mathfrak{m}_0$. Since H is the isotropy group of $([K], \nu_0)$, $\mathrm{Ad}_h(\nu_0) = \nu_0$ and this reduces to $[\nu_0, \mathfrak{n}] = \mathfrak{m}_0$. If we also consider the fact that N is rank one, then on $\mathfrak{m}$, $\ker(\mathrm{ad}_{\nu_0}|_{\mathfrak{m}}) = \mathfrak{s}$. Hence, $\ker(\mathrm{ad}_{\nu_0}) = \mathfrak{h} \oplus \mathfrak{s} = \mathfrak{l}$. The restriction $\mathrm{ad}_{\nu_0}|_{\mathfrak{p}}$ is thus injective and so we can deduce

$$\mathfrak{q} = \mathfrak{n} \oplus \mathfrak{m}_{\mathfrak{o}} = [\nu_0, \mathfrak{m}_0] + [\nu_0, \mathfrak{n}].$$

This operator ad_{ν_0} is of considerable significance for our understanding of the unit sphere bundle and manifold of geodesics. Not only does it provide the above isometry between the horizontal and vertical (with respect to $d\pi_N$) components of $\mathfrak{q}$, but as we shall see it also describes the symplectic structure on both manifolds and their compatible complex structures.

Lemma 2.3.3. *When restricted to the unit tangent bundle UN, the canonical one-form θ is of the form*

$$\theta(X_{[gH]}) = \left\langle \mathrm{Ad}_g(\nu_0), \beta_H(X_{[gH]}) \right\rangle.$$

The canonical symplectic form $\lambda|_{UN}$ is given by

$$\lambda(X, Y)|_{[gH]} = \langle [\mathrm{Ad}_g(\nu_0), \beta_H(X)], \beta_H(Y) \rangle.$$

Proof. If we let β_K denote the restriction $\beta_K : UN \to [\mathfrak{um}]_H$, then by the definition (9), for $X \in UN,\ Y \in T_X UN$,

$$\theta_X(Y) = \langle \beta_K(X), \beta_K(d\pi_N(Y)) \rangle = \langle \beta_K(X), \beta_H(Y) \rangle,$$

(where we've noted that in $\mathfrak{g}$, $\beta_K(d\pi_N(Y))$ is simply the $[\mathfrak{m}]_K$ component of $\beta_H(Y)$). By (8), we can locally choose a frame $F : UN \to G$ such that

$\beta_K(X) = \mathrm{Ad}_F(\nu_0)$. As with (6), this then allows us to describe β_H in terms of the pullback of the Maurer-Cartan form, $\alpha = F^*\omega$. Hence,

$$\theta(\cdot) = \langle \mathrm{Ad}_F(\nu_0), \mathrm{Ad}_F(\alpha_{\mathfrak{p}}(\cdot)) \rangle = \langle \nu_0, \alpha(\cdot) \rangle .$$

By the Maurer-Cartan equation $d\omega + [\omega, \omega] = 0$, since ν_0 is constant:

$$d\theta(Y, Z) = \langle \nu_0, [\alpha(Z), \alpha(Y)] \rangle$$

Since $\langle \cdot, \cdot \rangle$ is bi-invariant,

$$\langle \nu_0, [\mathfrak{h}, \mathfrak{p}] \rangle = - \langle [\nu_0, \mathfrak{h}], \mathfrak{p} \rangle .$$

We can also recall that $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$. Since $\mathfrak{h} \subset \ker \mathrm{ad}_{\nu_0}$, we are left with

$$\lambda(Y, Z) = \langle \nu_0, [\alpha_{\mathfrak{p}}(Z), \alpha_{\mathfrak{p}}(Y)] \rangle .$$

□

The symplectic form $\lambda_{\mathcal{Q}}$ is then derived from the restriction of this form. In order to simplify the notation of these forms, we can use the metric's left-invariance to equate them pointwise with forms at the basepoint. We can thus view $\lambda_{\mathcal{Q}}$ as the left-invariant 2-form corresponding to

$$\lambda_0 : \mathfrak{q} \times \mathfrak{q} \to \mathbb{R}; \ \lambda_0(\xi, \eta) = \langle [\nu_0, \xi], \eta \rangle .$$

2.4 The canonical complex structure on the manifold of geodesics

We shall now construct a complex structure on $\mathcal{Q}$. As we've seen, ad_{ν_0} is a bijective map on $\mathfrak{q} = [\nu_0, \mathfrak{g}]$. However, since $\mathrm{ad}_{\nu_0} : \mathfrak{m}_0 \longleftrightarrow \mathfrak{n}$, it can have no real eigenvalues. If we instead consider the complexification $\mathfrak{q}^{\mathbb{C}} = \mathfrak{q} \oplus i\mathfrak{q}$, then

$$\mathrm{ad}_{\nu_0}(\mathfrak{m}_0 + i\mathfrak{n}) = \mathfrak{n} + i\mathfrak{m}_0,$$

and so ad_{ν_0} has a set of non-zero eigenvalues iq_j such that $q_j \in \mathbb{R}$. The normal metric extends to the complexification by complex conjugation as $\langle \xi, \bar{\eta} \rangle$. Hence by skew-symmetry of the adjoint action, given an eigenvector $\xi_j \in \mathfrak{q}^{\mathbb{C}}$ of ad_{ν_0} with eigenvalue iq_j:

$$iq_j = \left\langle [\nu_0, \xi_j], \bar{\xi}_j \right\rangle = -\left\langle [\nu_0, \bar{\xi}_j], \bar{\bar{\xi}}_j \right\rangle,$$

and so $\bar{\xi}_j$ is also an eigenvector with eigenvalue $-iq_j$. If we then consider $\sigma_0 := \mathrm{ad}_{\nu_0}|_{\mathfrak{q}}$, we can observe that for an eigenvector X_j of ad_{ν_0}:

$$\sigma_0^2(\mathrm{Re}(\xi_j) = \frac{1}{2}\sigma_0^2(\xi_j) + \frac{1}{2}\sigma_0^2(\bar{\xi}_j) = -q_j^2.$$

We can thus decompose $\mathfrak{q}$ into ad_{ν_0}-invariant eigenspaces

$$\mathfrak{q} = \sum_j \mathfrak{q}_j; \;\; \sigma_0^2(X_j) = -q_j^2 X_j \text{ for all } X_j \in \mathfrak{q}_j.$$

Given a vector $\xi \in \mathfrak{q}$, if we let ξ_j denote the projection onto the eigenspace $\mathfrak{q}_j$, we can thus define

$$J_0 : \mathfrak{q} \to \mathfrak{q}; \;\; \xi \mapsto \frac{1}{q_j}\sigma_0(\xi_j).$$

Since $\mathfrak{l}$ is the centraliser of $\mathfrak{s}$, J_0 is Ad_L equivariant and so we can extend this to an almost complex structure J on $\mathcal{Q}$ such that $J\beta_L(X) = ([gL], \mathrm{Ad}_g(J_0\xi))$. To see that this almost complex structure is integrable, we use the following result.

Lemma 2.4.1. *[2, Proposition 8.39] Let G be a compact Lie group acting on its Lie algebra $\mathfrak{g}$ by the adjoint representation. Then the canonical G-invariant almost complex structure on $G/\mathrm{Stab}(o)$ for a basepoint $o \in \mathfrak{g}$ is integrable.*

Since $L = \mathrm{Stab}(\nu_0)$, this holds for $\mathcal{Q} = G/L$.

Lemma 2.4.2. *The canonical complex structure J on $\mathcal{Q}$ is an isometry with respect to the normal metric.*

Proof. Let $\xi, \eta \in \mathfrak{q}$. Since ad_{ν_0} is skew-symmetric:

$$\begin{aligned}
\langle J_0\xi, J_0\eta\rangle &= \sum_{j,k}\left\langle \frac{1}{q_j}[\nu_0, \xi_j], \frac{1}{q_k}[\nu_0, \eta_k]\right\rangle \\
&= -\sum_{j,k}\frac{1}{q_j q_k}\left\langle \sigma_0^2(\xi_j), \eta_k\right\rangle \\
&= -\sum_{j,k}\frac{1}{q_j q_k}\left\langle \xi_j, \sigma_0^2(\eta_k)\right\rangle .
\end{aligned}$$

Hence, either $q_j^2 = q_k^2$ or $\langle \xi_j, \eta_k\rangle = 0$. The eigenspaces $\mathfrak{q}_j$ are thus orthongonal and J_0 is an isometry. □

2.5 Metrics on the unit tangent bundle

While we have seen that the complex structure J is an isometry with respect to the normal metric on $\mathcal{Q}$, the normal metric on UN (hereby referred to as h_n) is not necessarily compatible with $\lambda_{\mathcal{Q}}$ and J in the manner required for a Kähler structure. To this end, we shall construct a new Kähler metric on $\mathcal{Q}$,

$$g_{\mathcal{Q}}(X, Y) := \lambda(X, JY).$$

The restriction of this metric to $\mathfrak{q}$ thus takes the form

$$g_{\mathcal{Q}}|_{\mathfrak{q}}(\xi, \eta) = \langle \nu_0, [\xi, J_0\eta]\rangle . \tag{10}$$

A particularly significant property of this metric is that it is Kähler-Einstein, as we shall see from the new few results.

Definition 2.5.1. *A Riemannian manifold (N, g) is an* Einstein *manifold if the Ricci curvature*

$$\mathrm{Ric}(X, Y) := \mathrm{tr}_g(R(\cdot, X)Y, \cdot)$$

is proportional to g. If N is also a Kähler manifold (N, ω, J) such that the Kähler structure is compatible with g, then (N, ω, J) is a Kähler-Einstein *manifold.*

Lemma 2.5.2. *[2, Proposition 8.85] If the centre $\mathfrak{c}$ of $\mathfrak{l}$ is one-dimensional, then G/L admits, up to scale, only one G-invariant Kähler structure. This structure coincides with the canonical symplectic structure λ_{can}, and $(G/L, \lambda_{can}, J)$ is Kähler-Einstein.*

To see how this applies to $\mathcal{Q}$, consider $\mathfrak{l} = \mathfrak{h} \oplus \mathfrak{s}$. Since $\mathfrak{l}$ is the centraliser of $\mathfrak{s}$, we already know that $\mathfrak{c}$ is at least one-dimensional. We must therefore consider $\mathfrak{h}$. To find an explicit form for $\mathfrak{h}$, we use the fact that for a CROSS, our isotropy groups K act transitively on U_oN. By the classification of transitive actions on spheres in [2]

$$\begin{aligned} US^n &\cong SO(n+1)/SO(n-1) \\ U\mathbb{CP}^n &\cong SU(n+1)/S(U(n-1) \times S^1) \\ U\mathbb{HP}^n &\cong Sp(n+1)/Sp(n-1) \times Sp(1) \\ U\mathbb{C}a\mathbb{P}^2 &\cong F_4/\mathrm{Spin}(7). \end{aligned}$$

For all of these except $\mathbb{CP}^n$, the corresponding Lie algebras ($\mathfrak{so}(n-1)$, $\mathfrak{sp}(n-1) + \mathfrak{sp}(1)$, $\mathfrak{so}(7)$) are semisimple and thus have trivial centre, so Lemma 2.5.2 applies.

In the case of $\mathbb{CP}^n$, we will need to consider the *Ricci form*,

$$\rho(X,Y) := \mathrm{Ric}(JX, Y).$$

Lemma 2.5.3. *[2, Corollary 8.59] Let G be a compact Lie group acting on its Lie algebra $\mathfrak{g}$ by the adjoint representation. The quotient $G/\mathrm{Stab}(o) = G/L$ for a basepoint o admits a Kähler-Einstein metric compatible with the complex structure. If $\{E_j, J_0E_j\}$ is an orthonormal basis for $\mathfrak{q}$ which consists of eigenvectors of the adjoint action of the centre $\mathfrak{c}$, then the Kähler form (up to scale) is given by the G-invariant form corresponding to*

$$\rho_0(\xi, \eta) = \left\langle \sum_j [E_j, J_0E_j], [\xi, \eta] \right\rangle.$$

The sum $\sum_j [E_j, J_0E_j]$ is indepenent of the choice of basis.

In the case that $\sum_j [E_j, J_0 E_j] = c\nu_0$, we can observe that $\rho_0(\xi, J_0\eta) = ch_{\mathcal{Q}}(\xi, \eta)$, and so $h_{\mathcal{Q}}$ is the unique (up to scale) Kähler-Einstein metric on $\mathcal{Q}$.

Lemma 2.5.4. *For $G = SU(n+1)$, $\rho_0 = \langle 2n\nu_0, [\cdot,\cdot]\rangle$. Hence, $(\mathcal{Q}, \lambda_{\mathcal{Q}}, J)$ is a Kähler-Einstein manifold.*

Proof. We can choose the basepoint $o \in \mathbb{CP}^n$ to be the point $[0, ..., 0, 1] = \{(0, ..., 0, z) : z \in \mathbb{C}\}$. The Lie algebra $\mathfrak{u}(n)$ consists of all skew-hermitian $n \times n$-matrices and $\mathfrak{su}(n)$ consists of all traceless skew-hermitian matrices. The reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ corresonding to $\mathbb{CP}^n \cong SU(n+1)/S(U(n) \times U(1))$ therefore takes the form

$$\mathfrak{k} = \left\{ \begin{pmatrix} A & 0 \\ 0 & a \end{pmatrix} : A \in \mathfrak{su}(n),\ a = -\mathrm{tr}(A) \in i\mathbb{R} \right\}$$

$$\mathfrak{m} = \left\{ \begin{pmatrix} 0 & u \\ -u^h & 0 \end{pmatrix} : u \in \mathbb{C}^n \right\},$$

where u^h is the Hermitian transpose. We can then choose our basepoint for G/H to be

$$\nu_0 = \begin{pmatrix} 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 1 \\ 0 & \dots & -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & e_n \\ -e_n^t & 0 \end{pmatrix},$$

where $\{e_1, ie_1, ..., e_n, ie_n\}$ is the standard basis for $\mathbb{C}^n$. It follows that

$$\mathfrak{n} = \mathrm{ad}_{\nu_0}(\mathfrak{m}_0) = \left\{ \begin{pmatrix} 0 & v & 0 \\ -v^h & ib & 0 \\ 0 & 0 & -ib \end{pmatrix} : v \in \mathbb{C}^{n-1},\ b \in \mathbb{R} \right\}$$

$$\mathfrak{q} = \mathfrak{n} \oplus \mathfrak{m}_0 = \left\{ \begin{pmatrix} 0 & v & w \\ -v^h & ib & ic \\ -w^h & ic & -ib \end{pmatrix} : v, w \in \mathbb{C}^{n-1},\ b, c \in \mathbb{R} \right\}.$$

By matrix multiplication with ν_0 as above, the eigenspace decomposition with respect to $\mathrm{ad}^2_{\nu_0}$ takes the form

$$\mathfrak{q}_1 = -\mathrm{ad}^2_{\nu_0}(\mathfrak{q}_1) = \left\{ \begin{pmatrix} 0 & v & w \\ -v^h & 0 & 0 \\ -w^h & 0 & 0 \end{pmatrix} : v, w \in \mathbb{C}^{n-1} \right\}$$

$$\mathfrak{q}_2 = -\frac{1}{4}\mathrm{ad}^2_{\nu_0}(\mathfrak{q}_2) = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & ib & ic \\ 0 & ic & -ib \end{pmatrix} : b, c \in \mathbb{R} \right\},$$

with eigenvalues $q_1^2 = -1$ and $q_2^2 = -4$ respectively. We can thus choose a basis of eigenvectors $\{E_j, J_0E_j\}$ as

$$\left\{ \begin{pmatrix} 0 & 0 & v \\ 0 & 0 & 0 \\ -v_j^h & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & v & 0 \\ -v_j^h & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : v_j \in \{e_1, ..., e_{n-1}, ie_1, ..., ie_{n-1}\} \right\}$$

for $\mathfrak{q}_1$ and

$$\left\{ E_{2n-1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix}, J_0E_{2n-1} \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix} \right\}$$

for $\mathfrak{q}_2$. By matrix multiplication again we can calculate

$$[E_j, J_0E_j] = \begin{cases} \nu_0, & j = 1, ..., 2n-2 \\ 2\nu_0, & j = 2n-1 \end{cases}$$

Hence, $\sum_j [E_j, J_0E_j] = 2n\nu_0$. □

Now that we've constructed the Kähler-Einstein metric $h_{\mathcal{Q}}$ on $\mathcal{Q}$, the question arises of whether or not it agrees with the normal metric $h_n|_{\mathcal{Q}}$. From [2, 8.86] we know that the normal metric is Kähler if and only if $(G/L, h_n)$ is itself a symmetric space. In the case of $\mathbb{CP}^n$ described above, we can observe that

$$\left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & ic \\ 0 & ic & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & w \\ 0 & 0 & 0 \\ -w^h & 0 & 0 \end{pmatrix} \right] = \begin{pmatrix} 0 & iaX & 0 \\ -iaX^h & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{q},$$

and therefore $[\mathfrak{q},\mathfrak{q}]$ is not contained in $\mathfrak{l}$. While h_n may not be Kähler for $\mathbb{CP}^n$, in the case of S^n, we can choose our decomposition so

$$\mathfrak{q} = \left\{ \begin{pmatrix} 0 & X & Y \\ -X^t & 0 & 0 \\ -Y^t & 0 & 0 \end{pmatrix} : X, Y \in \mathbb{R}^{n-1} \right\}, \tag{11}$$

and we can observe $[\mathfrak{q},\mathfrak{q}] \subseteq \mathfrak{l}$. Hence, for S^n, $\mathcal{Q}$ is a symmetric space (indeed, $\mathcal{Q} \cong \mathrm{Gr}(2, \mathbb{R}^{n+1})$) and $h_n|_{\mathcal{Q}} = h_{\mathcal{Q}}$.

Definition 2.5.5. *Let $UN = G/H$ be the unit tangent bundle of a CROSS with contact distribution $\mathcal{C} = \ker(d\theta)$ for the canonical one-form. Let Z_g denote the geodesic flow vector field with respect to the normal metric g on N. The lift of the Kähler-Einstein metric to UN is defined as*

$$h_{\mathcal{Q}}(X,Y) = \begin{cases} \lambda_{\mathcal{Q}}(d\pi_{\mathcal{Q}}(X), Jd\pi_{\mathcal{Q}}(Y)), & X, Y \in \mathcal{C} \\ h_n(X,Y), & X, Y || Z_g \\ 0, & X \in \mathcal{C}, Y || Z_g \end{cases}$$

We shall also lift the complex structure J on $\mathcal{Q}$ to $\mathcal{C} \subset UN$ such that for $X, Y \in \mathcal{C}$, $JX = Y$ when $Jd\pi_{\mathcal{Q}}X = d\pi_{\mathcal{Q}}Y$.

While the Kähler-Einstein metric is useful because of how it interacts with the symplectic and complex structures, another useful metric for the purpose of calculation, especially when $N = S^n$, is the Sasaki metric. Unlike $h_{\mathcal{Q}}$, this belongs to a family of metrics on UN which interacts nicely with the projection $d\pi_N$.

Lemma 2.5.6. *Let $(N, \langle\cdot,\cdot\rangle) \cong G/K$ be a CROSS with unit sphere bundle $UN \cong G/H$. Every G-invariant metric on G/H for which $\pi : UN \to N$ is a Riemannian submersion comes from an Ad_H-invariant inner product on $[\mathfrak{p}] = [\mathfrak{h}]^\perp$ of the form*

$$h(\xi,\eta) = \langle \xi_{\mathfrak{m}}, \eta_{\mathfrak{m}} \rangle + h_{\mathfrak{n}}(\xi_{\mathfrak{n}}, \eta_{\mathfrak{n}}),$$

where $h_{\mathfrak{n}}$ is any Ad_H-invariant metric on $\mathfrak{n}$. With respect to such a metric, π has totally geodesic fibres.

Proof. As shown in [6], the G-invariant metrics are in one-to-one correspondence with Ad_H-invariant inner products on $\mathfrak{p}$. Given such an inner product, h, the corresponding metric on G/H is such that

$$(X_{\pi(g)}, Y_{\pi(g)}) \mapsto h(dL_{g^{-1}}(X), dL_{g^{-1}}(Y)).$$

We will also denote this metric by h.

Since $[\mathfrak{m}]_H$ is the horizontal bundle for UN and N is equipped with the restriction of $\langle \cdot, \cdot \rangle$, it is clear that if π is a Riemannian submersion the metric must take the appropriate form. It remains only to show that π has totally geodesic fibres. In order to do this, we must first construct the Levi-Civita connection for h.

To construct the Levi-Civita connection ∇^h, we shall first construct a metric $\bar{h}$ on G such that $\pi_H : (G, \bar{h}) \to (G/H, h)$ is a Riemannian submersion. We define a left invariant metric on G from the inner product on $\mathfrak{g}$:

$$\bar{h}(\xi, \eta) = \langle \xi_{\mathfrak{h}+\mathfrak{m}}, \eta_{\mathfrak{h}+\mathfrak{m}} \rangle + h_{\mathfrak{n}}(\xi_{\mathfrak{n}}, \eta_{\mathfrak{n}}).$$

Using the left-invariance of the metric, we observe that the horizontal subbundle of TG with respect to π_H is the set $\mathcal{H} := \omega^{-1}(G \times \mathfrak{p})$, and so $\bar{h}$ agrees with h for horizontal vectors. Using Lemma 1.2.6 we have

$$\nabla^h_X Y = d\pi_H(\nabla^{\bar{h}}_{\bar{X}} \bar{Y}),$$

where $\bar{X}, \bar{Y}$ are the horizontal lifts of $X, Y \in \Gamma(TUN)$ (i.e. the unique vector fields in $\Gamma(\mathcal{H})$ such that $d\pi_H(\bar{X}_g) = X_{\pi_H(g)}$ for all $g \in G$).

To find an explicit form, we use Lemma 1.2.7, which states that for left-invariant vector fields X, Y on G:

$$\nabla^{\bar{h}}_X Y = \frac{1}{2}\left([X, Y] - (\mathrm{ad}_X)^*(Y) - (\mathrm{ad}_Y)^*(X)\right),$$

where $(\mathrm{ad})^*$ is the adjoint of ad. If we define left-invariant vector fields $E_1, ..., E_g$ which form a basis for $\mathfrak{g}$, then there exist functions $\xi^i, \eta^j : G \to \mathbb{R}$ such that

$\omega(\bar{X}) = \xi^i E_i, \omega(\bar{Y}) = \eta^j E_j$ (using the summation convention). We then calculate:

$$\begin{aligned}
\omega\left(\nabla^{\bar{h}}_{\bar{X}}\bar{Y}\right) &= \omega\left(\nabla^{\bar{h}}_{\xi^i\omega^{-1}(E_i)}\eta^j\omega^{-1}(E_j)\right) \\
&= \omega\left(\xi^i(\omega^{-1}(E_i))(\eta^j)E_j + \xi^i\eta^j\nabla^{\bar{h}}_{\omega^{-1}(E_i)}\omega^{-1}(E_j)\right) \\
&= \bar{X}\omega(\bar{Y}) + \xi^i\eta^j\nabla^{\bar{h}}_{E_i}E_j \\
&= \bar{X}\omega(\bar{Y}) + \frac{1}{2}\xi^i\eta^j\left([E_i, E_j] - \mathrm{ad}^*_{E_i}E_j - \mathrm{ad}^*_{E_j}E_i\right) \\
&= \bar{X}\omega(\bar{Y}) + \frac{1}{2}\left([\omega(\bar{X}), \omega(\bar{Y})] - \mathrm{ad}^*_{\omega(\bar{X})}\omega(\bar{Y}) - \mathrm{ad}^*_{\omega(\bar{Y})}\omega(\bar{X})\right).
\end{aligned}$$

Now that we have this form for the Levi-Civita connection, we can show that π has totally geodesic fibres. At the basepoint, the fibre is $S^{n-1} \cong K/H$. If we let $X, Y \in \Gamma(TS^{n-1})$, then since they are vertical with respect to π, the corresponding $\omega(\bar{X}), \omega(\bar{Y})$ take values in $\mathfrak{n}$. Since $[\mathfrak{n}, \mathfrak{n}] \subseteq \mathfrak{h}$, thus

$$\nabla^h_X Y = d\pi_H\left(\bar{X}\omega(\bar{Y}) - \frac{1}{2}\left(\mathrm{ad}^*_{\omega(\bar{X})}\omega(\bar{Y}) + \mathrm{ad}^*_{\omega(\bar{Y})}\omega(\bar{X})\right)\right).$$

Considering the adjoint terms, for $\xi \in \mathfrak{g}$:

$$\begin{aligned}
\bar{h}\left(d\pi_H\left(\mathrm{ad}^*_{\omega(\bar{X})}\omega(\bar{Y})\right), \xi\right) &= \bar{h}\left(\mathrm{ad}^*_{\omega(\bar{X})}\omega(\bar{Y}), \xi_{\mathfrak{p}}\right) \\
&= \bar{h}\left(\omega(\bar{Y}), [\omega(\bar{X}), \xi_{\mathfrak{p}}]\right) = 0,
\end{aligned}$$

since $[\mathfrak{n}, \mathfrak{p}] \subseteq \mathfrak{h} + \mathfrak{m} = \mathfrak{n}^\perp$. Hence, since $\omega(\bar{Y})$ is a map into $\mathfrak{n}$:

$$\nabla^h_X Y = d\pi_H(\bar{X}\omega(\bar{Y})) \in \Gamma(TS).$$

By left-invariance, we can observe that this is true for the other fibres. Thus, π has totally geodesic fibres with respect to $\mathfrak{h}$. □

Remark 2.5.7. A more general method to generate such metrics is given by [29]. They show that given a G-bundle $\pi : X \to (B, g)$, any associated bundle F with an Ehresmann connection and G-invariant metric generates a Riemannian metric on X for which π is a Riemannian submersion with totally geodesic fibres.

Given a horizontal and vertical splitting $T(TN) = \mathcal{H} \oplus \mathcal{V}$ such that $\mathcal{V} = \ker(d\pi_N)$, there exist isomorphisms between the vector spaces $\mathcal{H}_X, \mathcal{V}_X$ and

$T_{\pi_N(X)}(N)$. The horizontal identification is given by $d\pi_N(\mathcal{H}_X) = T_{\pi_N(X)}N$. The vertical identification is the standard identification between the vector space $T_{\pi_N(X)}$ and its own tangent space $T(T_{\pi(X)}N)$.

Let $\kappa : TTN \to TN$ be the connector associated with the Levi-Civita connection on (N, g) such that $\nabla_X Y = \kappa(dY(X))$ (the differential $dY : TN \to TTN$ is defined by viewing Y as a smooth map between manifolds $Y : N \to TN$). If we let $X^{\mathcal{V}}$ denote the projection of X onto the vertical subbundle $\mathcal{V}$, the Sasaki metric takes the form

$$h_S(X, Y) = g(d\pi_N(X), d\pi_N(Y)) + g(\kappa(X^{\mathcal{V}}), \kappa(Y^{\mathcal{V}})).$$

The inclusion of $UN \subset TN \cong [\mathfrak{m}]_K$ takes the form

$$\nu : G/H \to [\mathfrak{m}]_K;\ gH \mapsto ([gK], \mathrm{Ad}_g(\nu_0)). \tag{12}$$

At the basepoint, horizontal vectors $\xi_{\mathfrak{m}}$ can be identified with vectors parallel to $e^{t\xi_{\mathfrak{m}}}[K]$, and vertical vectors are $\xi_{\mathfrak{n}}$ are tangent to the curve $e^{t\xi_{\mathfrak{n}}}[H]$. Hence, the differential of this map is given by

$$(d\nu)_{\mathfrak{o}} : T_{[H]}(G/H) \cong \mathfrak{p} \to T\mathfrak{m} \cong \mathfrak{m} + \mathfrak{m};\ \xi \mapsto \xi_{\mathfrak{m}} + [\xi_{\mathfrak{n}}, \nu_0].$$

The Sasaki metric thus corresponds to taking the normal metric on each copy of $\mathfrak{m}$.

Definition 2.5.8. *The (restriction of the) Sasaki metric h_s on UN is the G-invariant metric corresponding to the metric on $\mathfrak{p}$ given by*

$$\mathfrak{p} \times \mathfrak{p} \to \mathbb{R};\ (\xi, \eta) \mapsto \langle \xi_{\mathfrak{m}}, \eta_{\mathfrak{m}} \rangle + \langle [\nu_0, \xi_{\mathfrak{n}}], [\nu_0, \eta_{\mathfrak{n}}] \rangle .$$

While the Sasaki metric has useful properties, in general it doesn't descend to a metric on $\mathcal{Q}$. In the case of a sphere $N = S^n$, matrix multiplication in the decomposition (11) tells us $\mathrm{ad}_{\nu_0}^2 = -id$, and so by the self-adjointness of ad_{ν_0}, $h_s = h_n = h_{\mathcal{Q}}$. In general it is not possible to define a restriction of h_s to $\mathcal{Q}$, as the following result proves.

Proposition 2.5.9. *Let (N, g) be a complete Riemannian manifold. Let (UN, h_s) be the unit tangent bundle equipped with the restriction of the Sasaki metric. The geodesic flow acts on UN by isometries if and only if N has constant sectional curvature equal to* 1*.*

Proof. Let $v \in TN$. We will denote by $\gamma_v : \mathbb{R} \to N$ the geodesic with initial condition $\dot{\gamma}_v(0) = v$. The geodesic flow $\Phi : \mathbb{R} \times TN \to TN$ is the family of diffeomorphisms

$$\Phi(t, \cdot) = \varphi^t : TN \to TN; \quad \varphi^t(v) = \dot{\gamma}_v(t)$$
$$\Phi(\cdot, v) = c_v : \mathbb{R} \to TN \quad ; \quad c_v(t) = \dot{\gamma}_v(t).$$

Given a unit vector $u \in UN$, $g(\dot{\gamma}_u(t), \dot{\gamma}_u(t)) = g(u, u) = 1$, so we can restrict the flow to UN.

Given a vector $\xi \in T_u UN$, we can generate a Jacobi field J_ξ along the geodesic $\gamma_u(t)$ with initial conditions

$$J_\xi(0) = d\pi_N(\xi)$$
$$D_t\big|_{t=0} J_\xi = \kappa_u(\xi).$$

From this Jacobi field we can define a section $\mathcal{J}_\xi \in c_u^{-1}(TUN)$ by

$$\mathcal{J}_\xi(t) = (d\pi_N|_{\mathcal{H}})^{-1}_{c_u(t)}(J_\xi(t)) + (\kappa_{c_u(t)}|_{\mathcal{V}})^{-1}(D_t J_\xi(t)),$$

where

$$\overline{J_\xi(t)} := (d\pi_N|_{\mathcal{H}})^{-1}_{c_u(t)}(J_\xi(t)),$$
$$\mathrm{vl}_{c_u(t)}(J_\xi(t)) := (\kappa_{c_u(t)}|_{\mathcal{V}})^{-1}(D_t J_\xi(t))$$

are the horizontal lift and vertical lift through $c_u(t)$ of $J_\xi(t)$ (note that $d\pi_N$ restricted to the horizontal bundle and κ restricted to the vertical bundle are isometries, so this is well defined).

Since J_ξ is a Jacobi field, there exists a geodesic variation $a(t, s)$ of γ_u such that $J_\xi(t) = \frac{\partial}{\partial s}\big|_{s=0} a(t, s)$. For a fixed s, $a(\cdot, s) : \mathbb{R} \to N$ is a geodesic, so for some curve $X : \mathbb{R} \to TN$:

$$a(t,s) = (\pi_N \circ \varphi^t)(X(s)).$$

We would like to choose this curve to be such that $\frac{\partial}{\partial s}\big|_{s=0} X(s) = \xi$, but must first check that such a curve fits with the initial conditions for J_ξ. We first note that

$$\frac{\partial}{\partial s}\Big|_{s=0} a(0,s) = \frac{\partial}{\partial s}\Big|_{s=0} \pi_N(X(s)) = d\pi_N(\xi) = J_\xi(0).$$

For the second condition, we consider

$$D_t\big|_{t=0} J_\xi(t) = \nabla_{c_u(0)} J_\xi(t).$$

Noting that $[c_u, J_\xi] = \left[da(\frac{d}{dt}), da(\frac{d}{ds})\right] = 0$, this becomes

$$\begin{aligned} D_t\big|_{t=0} J_\xi(t) &= \nabla_{J_\xi(0)} c_u(s) \\ &= D\big|_{s=0} \frac{\partial}{\partial t}\Big|_{t=0} a(t,s) \\ &= D\big|_{s=0} d\pi_N \left(\frac{\partial}{\partial t}\Big|_{t=0} c_{X(s)}(t) \right). \end{aligned} \tag{13}$$

Differentiating the geodesic flow, we obtain the geodesic vector field Z. Since this is the Reeb vector field for the canonical 1-form λ on TN:

$$d\pi_N(Z_{X(s)}) = \pi_{TN}(Z_{X(s)}) = X(s).$$

We thus have

$$\begin{aligned} D_t\big|_{t=0} J_\xi(0) &= D_s\big|_{s=0} X(s) \\ &= \nabla_{\frac{\partial}{\partial s}}\big|_{s=0} X(s) \\ &= \kappa\left(dX_0 \left(\frac{\partial}{\partial s} \right) \right) = \kappa(\xi). \end{aligned}$$

Hence, our geodesic variation $a(t,s)$ is compatible with J_ξ. By using the form

$$J_\xi(t) = \frac{\partial}{\partial s}\Big|_{s=0} a(t,s) = d(\pi_N \circ \varphi^t)_u(\xi),$$

we can find a convenient form for the differential of φ^t. Consider (13) without evaluating at $t = 0$:

$$\begin{aligned}
D_t J_\xi(t) &= D_s\big|_{s=0} d\pi_N \left(\frac{\partial}{\partial t} c_{X(s)}(t)\right) \\
&= D_s\big|_{s=0} d\pi_N (Z_{\varphi^t(X(s))}) \\
&= D_s\big|_{s=0} \varphi^t(X(s)) \\
&= \kappa \left(\left(d\varphi^t(X)\right)_0 \left(\frac{\partial}{\partial s}\right)\right) = \kappa\left(d\varphi^t_u(\xi)\right).
\end{aligned}$$

We can thus easily observe

$$\begin{aligned}
\mathcal{J}_\xi(t) &= (d\pi_N|_{\mathcal{H}})^{-1} \left(d\pi_N \left(d\varphi^t_u(\xi)\right)\right) + (\kappa|_{\mathcal{V}})^{-1} \left(\kappa\left(d\varphi^t_u(\xi)\right)\right) \\
&= \left(d\varphi^t_u(\xi)\right)^{\mathcal{H}} + \left(d\varphi^t_u(\xi)\right)^{\mathcal{V}} = d\varphi^t_u(\xi).
\end{aligned}$$

To check if the geodesic flow acts by isometries, we can thus consider

$$\begin{aligned}
\frac{d}{dt} h_s \left(d\varphi^t_u(X), d\varphi^t_u(X)\right) &= \frac{d}{dt}\left(g(J_\xi(t), J_\xi(t)) + g(D_t J_\xi(t), D_t J_\xi(t))\right) \\
&= 2\left(g\left(J_\xi(t), D_t J_\xi(t)\right) + g\left(D_t J_\xi(t), D_t^2 J_\xi(t)\right)\right).
\end{aligned}$$

Since J_ξ is a Jacobi field,

$$D_t^2 J_\xi = -R(J_\xi, c_u)c_u.$$

Thus, the geodesic flow acts by isometries on UN if and only if for all $u \in TUN$, $\xi \in T_uUN$, $t \in \mathbb{R}$:

$$g\left(J_\xi(t) - R(J_\xi(t), c_u(t))c_u(t), D_t J_\xi(t)\right) = 0. \tag{14}$$

Let us assume that the geodesic flow acts by isometries on UN. Let X, Y be orthogonal unit vectors at a point $p \in N$. Given $t \in \mathbb{R}$, define $u = \varphi^{-t}(X)$. Then, $X = c_u(t)$. We can define a vector $\xi \in T_uUN$ by

$$\xi = d\varphi_X^{-t}\left(\bar{Y} + \mathrm{vl}_X(Y)\right).$$

This vector ξ has the property $J_\xi(t) = D_t J_\xi(t) = Y$. Hence by (14)

$$g\left(Y - R(Y, X)X, Y\right) = 0 = 1 - K(Y, X).$$

Since X, Y were arbitrary, we have $K = 1$.

If we instead assume $K = 1$, then for any vector fields ξ, η along γ_u,

$$g\left(R(\xi, D_t\gamma_u)D_t\gamma_u, \eta\right) = g(\xi, \eta)$$

and so $\frac{d}{dt}h_s$ vanishes. $\square$

3 The geodesic Gauss map

3.1 Harmonic maps and the tension field

We shall be concerning ourselves with two related objects: harmonic maps and minimal submanifolds. Both objects involve the local minimisation of energy, namely Dirichlet energy for harmonic maps and volume for minimal submanifolds. The more immediately concrete of the two, growing out of surface theory, is the concept of minimality.

Given an isometrically immersed submanifold $f : M \to (N, g, \nabla)$, the *second fundamental form* of f is given by

$$\mathrm{II}_f \in \Gamma(T^*M \otimes T^*M \otimes TM^\perp); \quad \mathrm{II}_f(X, Y) = (\nabla^f_X(df(Y)))^\perp$$

It serves to relate the curvature of M to the curvature of N, and when $\mathrm{II}_f = 0$ we say M is *totally geodesic* since the image of a geodesic of M is itself a geodesic of N (for example, the totally geodesic submanifolds of Euclidean space are precisely the subplanes). Minimality comes from a somewhat weaker condition.

Definition 3.1.1. *Let $f : M \to (N, g, \nabla)$ be an isometrically immersed submanifold. The* mean curvature *of f is given by*

$$H_f \in \Gamma(TM^\perp); \quad H_f(p) = \mathrm{tr}_g \mathrm{II}_f.$$

We say that f is minimal *if $H_f \equiv 0$*

In some conventions the mean curvature is instead defined as $\frac{1}{\mathrm{dim(M)}} \mathrm{tr}_g \mathrm{II}_f$, but they still have the same condition for minimality and since for our submanifolds $dim(M)$ is a constant it is easier for us to ignore this term.

To see how H_f relates to curvature, we can consider the *shape operator*

$$A_f \in \Gamma(\mathrm{Hom}(TM^\perp, \mathrm{End}(TM, TM))); \quad f^*g(A_f(\eta)X, Y) = g(\mathrm{II}_f(X, Y), \eta). \quad (15)$$

In the case of an oriented hypersurface, the eigenvalues κ_i of the shape operator are the principal curvatures of M and thus $||H_f|| = |\sum_{i=1}^m \kappa_i|$.

When considering harmonic maps, we start from the more general position of a smooth map $\varphi : (M, g) \to (N, h)$ between Riemannian manifolds. We define the *energy density* of φ to be

$$e(f) = \frac{1}{2} \langle d\varphi, d\varphi \rangle ,$$

where $\langle \cdot, \cdot \rangle$ is the induced metric on $T^*M \otimes \varphi^{-1}(TN)$.

Definition 3.1.2. *Given a smooth map $\varphi : (M, g, \nabla^M) \to (N, h, \nabla^N)$, φ is a* harmonic map *if it is a critical point of the* energy

$$E(\varphi) = \int_M e(\varphi) d\mathrm{Vol}_g.$$

As shown in [15], φ is a harmonic map if and only if, for the induced connection on $T^*M \otimes \varphi^{-1}(TN)$, $\mathrm{tr}_g \nabla df \equiv 0$. We thus define the *tension field*

$$\tau(f) \in \Gamma(f^{-1}TN); \quad \tau(f) = \mathrm{tr}_g \nabla df.$$

We shall be interested in the case where φ is an immersion. In this case,

$$\nabla d\varphi(X, Y) = \nabla_X^{\varphi}(d\varphi(Y)) - d\varphi(\nabla_X^M Y).$$

We can see that this object functions as a generalisation of the second fundamental form, measuring the difference between the connection on M and the ambient connection on N. In fact when φ is an isometric immersion this difference is simply the normal component of the ambient connection, and thus $\nabla d\varphi(X, Y) = \mathrm{I\!I}_{\varphi}(X, Y)$. Hence, an isometric immersion is minimal if and only if it is a harmonic map.

3.2 The Ruh-Vilms theorem

With these concepts in mind, we can now turn to the theorem we shall be expanding on, the Ruh-Vilms theorem [25]. The classical Gauss map, $\hat{\gamma}$, for a hypersurface $f : M \to \mathbb{R}^{n+1}$ is given by identifying the unit tangent space U_pN with the unit sphere $S^n \subset \mathbb{R}^{n+1}$ and assigning to the point p the (oriented) unit normal vector in $U_pM^\perp \subset U_{f(p)}\mathbb{R}^{n+1}$. There are multiple methods of expanding this definition to other spaces. The first, as used by Ruh and Vilms, is to view the classical Gauss map as instead assigning to the point p the subspace $df(T_pM) \subset T_{f(p)}\mathbb{R}^{n+1} \cong \mathbb{R}^{n+1}$. This now extends to submanifolds of Euclidean space of higher codimension, in which case the Gauss map is now a map into the Grassmannian manifold $\mathrm{Gr}(m, n+1)$ of $m = \dim(M)$-dimensional subspaces of $\mathbb{R}^{n+1}$.

In [25], Ruh and Vilms consider an m-dimensional isometrically immersed submanifold $f : M \to \mathbb{R}^n$. They construct an isomorphism $\hat{\gamma}_f^{-1}\mathrm{T}Gr(m,n) \cong \mathrm{Hom}(TM, TM^\perp)$. Since the mean curvature vector field for f takes values in $TM^\perp$, we see $\nabla^\perp H_f \in \mathrm{Hom}(TM, TM^\perp)$. Using this identification, they then prove the following result.

Theorem 3.2.1 (Ruh-Vilms)**.** *For an isometrically immersed submanifold $f : M \to \mathbb{R}^n$ with Gauss map $\hat{\gamma}_f$:*

$$\tau(\hat{\gamma}_f) = \nabla^\perp H_f.$$

The immediate implication is that the mean curvature vector field for f is parallel if and only if $\hat{\gamma}_f$ is a harmonic map. Since minimality implies parallel mean curvature vector field, we can observe that the classical Gauss map for any minimal submanifold of $\mathbb{R}^n$ is a harmonic map.

An obvious limitation of this result is that it only applies to submanifolds of Euclidean space. Several variants of the Gauss map have been used since to establish similar results under less restrictive conditions. In [21], Obata defines a *generalised Gauss map* for use on spaces of constant curvature. This assigns to a point the totally geodesic subspace generated by its tangent space. For example for a submanifold $f : M \to S^n \subset \mathbb{R}^{n+1}$ this would be the intersection of S^n with the plane $\hat{\gamma}_f(p)$. This version of the Gauss map was then used to show that a pseudo-umbilical immersion has a conformal Gauss map.

Wood's [28] approach was to instead consider an isometric immersion into a more general manifold N. By considering the bundle of orthonormal frames over N, the *Gauss section*, $\tilde{\gamma}_f$ can be defined as the equivalence class of frames adapted to the immersion (which in the relevant cases is equivalent to the previous definitions). By considering only the vertical component of the tension field, Wood found that when the Ricci curvature obeys the condition $f^*\mathrm{Ric}^\perp = 0$ (for example on space forms and for hypersurfaces in Einstein manifolds), then $\tau^{\mathcal{V}}(\tilde{\gamma}_f) = 0$ ($\tilde{\gamma}_f$ is a *harmonic section*) if and only if $\nabla H_f = 0$.

More recently, Jensen and Rigoli [14] defined the *spherical Gauss map* μ_f, which instead of being a map on M is a map on $UM^\perp$, which assigns to each unit normal vector its inclusion in UN. They were able to find conditions for μ_f to be harmonic with respect to the Sasaki metric induced by $\pi^\perp : UM^\perp \to M$ for a minimal isometric immersion, namely that $T = 0, f^*\mathrm{Ric}^\perp = 0$ and the second fundamental form is conformal, where the tensor $T \in TM^\perp \otimes TM^\perp \otimes f^{-1}(TN)$ is given by $T^{\alpha A}_\beta = h^\alpha_{ij} R^i_{\beta j A}$ where h^α_{ij} are the components of the second fundamental form and R is the Riemannian curvature.

In [7], Cintract and Morvan find another condition for the spherical Gauss map with respect to the Sasaki metric, namely that f is minimal if and only if μ_f is austere, meaning that for the eigenvalues of the shape form, $\sum_{i=1}^m \arctan\kappa_i \cong 0 \bmod \pi$.

We shall be trying something a bit different, by using different metrics to obtain symplectic properties of the manifold of geodesics.

3.3 The spherical and geodesic Gauss maps

Let $(N, g) = (G/K, \langle\cdot,\cdot\rangle)$ be a compact rank one symmetric space. Let $f : M \to N$ be an isometrically immersed submanifold. The *normal bundle* of M is the manifold

$$TM^{\perp} := \left\{(p, X) \in f^{-1}(TN) :\ g(X, df(T_pM)) = 0\right\}.$$

The *unit normal bundle* $UM^{\perp} \subset TM^{\perp}$ is the unit length subbundle of $TM^{\perp}$. We define $\pi^{\perp} : UM^{\perp} \to M$ to be the bundle projection.

Definition 3.3.1. *The* spherical Gauss map $\mu : UM^{\perp} \to UN$ *is the immersion induced by f such that $\pi_N \circ \mu = f \circ \pi^{\perp}$.*

If we consider the canonical one-form θ_{can} on $UM^{\perp}$, it clearly vanishes, and so $T(UM^{\perp}) \subset \mathcal{C} \subset TUN$. Since

$$\dim(UM^{\perp}) = \dim(M) + (\dim(N) - \dim(M)) - 1 = n - 1,$$

$UM^{\perp}$ is thus a Legendrian submanifold of UN.

Definition 3.3.2. *The* geodesic Gauss map *of an isometrically immersed submanifold $f : M \to N$ of a compact rank one symmetric space is given by*

$$\gamma_f : UM^{\perp} \to \mathcal{Q};\ \gamma_f = \pi_{\mathcal{Q}} \circ \mu.$$

The relationship between our various manifolds can be summed up by the following commutative diagram.

$$\begin{tikzcd}
 & G \arrow[d, "\pi_H"'] \arrow[ddr, bend left, "\pi_L"] & \\
UM^{\perp} \arrow[r, "\mu"] \arrow[d, "\pi^{\perp}"'] & UN \cong G/H \arrow[d, "\pi_N"'] \arrow[dr, "\pi_{\mathcal{Q}}"'] & \\
M \arrow[r, "f"] & N \cong G/K & \mathcal{Q} \cong G/L
\end{tikzcd}$$

We are going to study the tension field of the spherical Gauss map in order to obtain results regarding the harmonicity of the geodesic Gauss map. We will do this using the following result.

Lemma 3.3.3. *[8, Lemma 3.1] Let M, N, S be Riemannian manifolds. Suppose $\varphi : M \to N$ is a smooth immersion and $\psi : N \to S$ is a Riemannian submersion. If φ is horizontal, then $\tau(\varphi)$ is also horizontal and $\tau(\psi \circ \varphi) = \psi_* \tau(\varphi)$. Hence, φ is harmonic if and only if $\psi \circ \varphi$ is harmonic. Further, if φ is isometric then so is $\psi \circ \varphi$, therefore φ is minimal if and only if $\psi \circ \varphi$ is minimal.*

Proof. Let $\gamma = \psi \circ \varphi$. We can thus think of φ as the horizontal lift of γ. Since φ is a horizontal immersion, γ is also an immersion. Let E_i be a local orthonormal frame for M. Then using Lemma 1.2.6,

$$\begin{aligned} \psi_* \tau(\varphi) &= \sum_j \psi_* (\nabla^{\varphi}_{E_j} (d\varphi(E_j)) - d\varphi(\nabla^M_{E_j} E_j)) \\ &= \nabla^{\gamma}_{E_j} (d\gamma(E_j)) - d\gamma(\nabla^M_{E_j} E_j) = \tau(\gamma). \end{aligned}$$

The vertical component of $\tau(\varphi)$ is the vertical component of

$$\nabla^{\varphi}_{E_j} (d\varphi(E_j)) = \nabla^N_{d\varphi(E_j)} d\varphi(E_j),$$

but $\nabla^N_X X$ is horizontal for a horizontal lift. Hence, $\tau(\varphi)$ is horizontal. Finally, since ψ is Riemannian it will preserve the isometry. □

In this way, whenever $\pi_{\mathcal{Q}}$ is a Riemannian submersion, $\tau(\gamma)$ is determined by $\tau(\mu)$.

In cases where we are working with the Kähler-Einstein metric on $\mathcal{Q}$, minimality of μ further implies that $\gamma : UM^{\perp} \to \mathcal{Q}$ is *Lagrangian stationary.*

Definition 3.3.4. *Let (N, ω, g) be a Kähler manifold. Given a Lagrangian isometric immersion $\varphi : M \to (N, \omega, g)$, φ is* Lagrangian stationary *if it is a critical point of the volume*

$$V(\varphi) = \int_M d\mathrm{Vol}$$

with respect to variations by Lagrangian immersions.

To see that this applies to γ when μ is minimal, we use the following result.

Lemma 3.3.5. *[26, Lemma 8.2] Let $\varphi : M \to (N, \omega, g)$ be a Lagrangian immersion into a Kähler-Einstein manifold. Then φ is Lagrangian stationary if and only if it is minimal.*

In the case of a Kähler-Einstein manifold we thus also refer to Lagrangian stationary manifolds as *minimal Lagrangian*.

4 Harmonicity of the geodesic Gauss map over S^n

Much of the work in this chapter is also present in [8], a joint paper with my supervisor Ian McIntosh.

4.1 Frames for the unit normal bundle

We shall now restrict our attention to the special case of a sphere, $S^n \cong G/K$. As we have already seen, in this case our three metrics on UN agree, i.e. $h_s = h_n = h_{\mathcal{Q}}$. We shall thus equip UN with this metric and will commonly refer to it as the normal metric h_n. The submanifold $\mu : UM^\perp \to UN$ will then be equipped with the pullback metric $\mu^* h_n$ (which agrees with the pullback by γ of the metric on $\mathcal{Q}$, since $\pi_{\mathcal{Q}}$ is a Riemannian submersion). Since we've assumed M is isometrically immersed it is equipped with the pullback metric f^*g, where as usual g is the normal metric on G/K. Since h_n is of the form described in Lemma 2.5.6, $\pi_N : UN \to N$ is clearly a Riemannian submersion with totally geodesic fibres with respect to these metrics.

In order to work on these manifolds we shall want to use local frames into $G = SO(n+1)$. We first choose open contractible neighbourhoods $U \subset M$, $V \subset UM^\perp$ such that $\pi^\perp(V) = U$ and equip V with a local frame $\Phi : V \to G$ over μ. Since $\pi_N \circ \mu = f \circ \pi^\perp$, we can define a frame $F : U \to G$ such that

$$(F \circ \pi^\perp)K = \Phi K. \tag{16}$$

As with (6), we can use the Maurer-Cartan form to define $\mathfrak{g}$-valued 1-forms associated with the frames. We denote these by $\varphi = \Phi^*\omega : TV \to \mathfrak{g}$ and $\alpha = F^*\omega : TU \to \mathfrak{g}$. To see how they relate, we consider the map $\Psi : V \to K \subset G$ defined by $\Phi = (F \circ \pi^\perp)\Psi$, and define the related form $\psi : V \to \mathfrak{k} \subset \mathfrak{g}$ by

$\psi = \Psi^{-1}d\Psi$. If we consider the forms as matrices:

$$\begin{aligned}\varphi = \Phi^{-1}d\Phi &= (F_{\pi^\perp}\Psi)^{-1}d(F_{\pi^\perp}\Psi)\\ &= \mathrm{Ad}_{\Psi}^{-1}\alpha_{d\pi^\perp} + \psi.\end{aligned}$$

Since ψ takes values in $\mathfrak{k}$, we can thus observe

$$\begin{aligned}\varphi_{\mathfrak{m}} &= \mathrm{Ad}_{\Psi}^{-1}(\alpha \circ d\pi^\perp)_{\mathfrak{m}}\\ \mathrm{Ad}_{\Phi}\varphi_{\mathfrak{m}} &= \mathrm{Ad}_F\alpha_{\mathfrak{m}}(d\pi_n). \qquad (17)\end{aligned}$$

From Corollary 1.2.11 (and noting that N is symmetric) we obtain expressions for the induced connections:

$$\begin{aligned}\beta_H(\nabla^\mu_X Y) &= \mathrm{Ad}_\Phi\left(X\varphi_{\mathfrak{p}}(Y) + [\varphi_{\mathfrak{h}}(X), \varphi_{\mathfrak{p}}(Y)] + \frac{1}{2}[\varphi_{\mathfrak{p}}(X), \varphi_{\mathfrak{p}}(Y)]_{\mathfrak{p}}\right) \qquad (18)\\ \beta_K(\nabla^f_X Y) &= \mathrm{Ad}_F\left(X\alpha_{\mathfrak{m}}(Y) + [\alpha_{\mathfrak{k}}(X), \alpha_{\mathfrak{m}}(Y)]\right).\end{aligned}$$

We shall also want to equip μ^*UN locally with an orthonormal moving frame of vectors adapted to the submanifold $UM^\perp$. In particular, we shall want this adapted frame to respect the horizontal bundle for $\pi^\perp$ induced by the connection on N. First we shall consider the horizontal bundle for $d\pi_N : TUN \to TN$.

Lemma 4.1.1. *Given a CROSS $(N \cong G/K, g, \nabla)$ equipped with the normal metric and its Levi-Civita connection, let $Z \in T_\xi UN$ have vertical component $Z^{\mathcal{V}} \in \mathcal{V} := \ker(d\pi_N) \cong [\mathfrak{n}]_H$. Let $\nu : G/H \to [\mathfrak{u}\mathfrak{m}]_H$ be the tautological normal section $\nu(gH) = (gH, \mathrm{Ad}_g\nu_0)$. Then*

$$d\pi_N([\beta_H(Z^{\mathcal{V}}), \nu]) = \beta_K(\nabla_{d\pi_N(Z)}Y),$$

where $Y(t) \subset UN$ is any curve satisfying $Y(0) = \xi$, $\frac{d}{dt}|_{t=0}Y(t) = Z$. The vector Z is horizontal whenever $Y(t)$ is parallel along $\pi_N(Y(t))$, and hence the horizontal bundle induced by ∇ is given by $\mathcal{H} \cong [\mathfrak{m}]_H$.

Proof. If we consider the curve $Y(t) \in G/H$, we can lift it to a curve $g(t) \in G$ such that $Y(t) = g(t)H$. As we saw earlier, when considering such a lift into G,

$\beta_H(X) = \mathrm{Ad}_g(\pi_G(\omega(\bar{X}))$. Hence, if we consider the vector $\eta = g^{-1}\dot{g}(0) \in \mathfrak{g}$,

$$\beta_H(Z) = \beta_H(\dot{Y}(0)) = \mathrm{Ad}_{g_0}\eta_{\mathfrak{p}},$$

And thus $\beta_H(Z^{\mathcal{V}}) = \mathrm{Ad}_{g_0}\eta_{\mathfrak{n}}$.

When we restrict β_K to UN, it takes the form $\beta_K(Y) = \mathrm{Ad}_{g_0}\nu_0 \in [\mathfrak{um}]_K$. Since M is a symmetric space, $\beta_K(\nabla_X Y) = P_{\mathfrak{m}}(X\beta_K(Y))$. For $d\pi_N(Z) = \frac{d}{dt}|_{t=0}(\pi_N \circ Y)$, this gives

$$\begin{aligned}
\beta_K(\nabla_{d\pi_N(Z)}Y) &= P_{\mathfrak{m}}\left(\frac{d}{dt}|_{t=0}\beta_K(Y(t))\right) \\
&= \mathrm{Ad}_{g_0}\left(\frac{d}{dt}|_{t=0}\nu_0 + \mathrm{ad}_\eta \nu_0\right)_{\mathfrak{m}} \\
&= \mathrm{Ad}_{g_0}[\eta, \nu_0]_{\mathfrak{m}}
\end{aligned}$$

Since $\mathrm{ad}_{\nu_0} : \mathfrak{m}_0 \longleftrightarrow \mathfrak{n}$, we can observe this reduces to

$$\beta_K(\nabla_{d\pi_N(Z)}Y) = \mathrm{Ad}_{g_0}([\eta_{\mathfrak{n}}, \nu_0]) = d\pi_N([\beta_H(Z^{\mathcal{V}}), \nu]).$$

The right hand side of this vanishes when $\beta_H(Z^{\mathcal{V}}) \in [\mathfrak{n}]_H$ vanishes, and thus the horizontal bundle coincides with $[\mathfrak{m}]_H$. $\square$

If we now turn our attention to $UM^\perp$, we shall denote the horizontal and vertical bundles for $\pi^\perp$ by $\mathcal{H}_M$ and $\mathcal{V}_M$ respectively. Clearly $\mathcal{V}_M = T(UM^\perp) \cap \mathcal{V}$, but it's not usually the case that $\mathcal{H}_M$ is contained in $\mathcal{H}$, as we shall soon see. To distinguish between the two we shall henceforth refer to horizontal vectors with respect to $\pi^\perp$ as $\pi^\perp$-*horizontal*.

The previous lemma does show us the form of $\mathcal{H}_M$, however. If we consider the canonical complex structure J on $\mathcal{C}$, as we have already seen, $UM^\perp$ is a Legendrian submanifold and thus J is well-defined over $UM^\perp$. In the case of a sphere, we also know $\mathrm{ad}^2_{\nu_0} = -1$, and thus J is the G-invariant tensor corresponding to ad_{ν_0}. Since J maps $[\mathfrak{m}_0]_H \longleftrightarrow [\mathfrak{n}]_H$, we can observe the following identities for any vector field Z in $\mathcal{C}$:

$$\varphi_{\mathfrak{n}}(JZ) = J_0\varphi_{\mathfrak{m}}(Z),$$
$$\varphi_{\mathfrak{m}}(JZ) = J_0\varphi_{\mathfrak{n}}(Z).$$

Lemma 4.1.2. *Let $f : M \to S^n$ be an isometrically immersed submanifold. Let $Z \in T_\xi(UM^\perp)$ be a $\pi^\perp$-horizontal vector. We denote $Z = Z^{\mathcal{H}} + Z^{\mathcal{V}}$ with respect to the the horizontal and vertical splitting $\mu^{-1}\mathcal{H} \oplus \mu^{-1}\mathcal{V}$ for $d\pi_N$. The splitting takes the form*

$$Z^{\mathcal{H}} = \overline{d\pi_N(Z)}$$
$$Z^{\mathcal{V}} = -J\overline{A_f(\xi)d\pi_N(Z)},$$

where A_f is the shape operator and $\bar{X}$ is the horizontal lift of X.

Corollary 4.1.3. *The horizontal bundles $\mathcal{H}_M$ and $\mathcal{H}$ agree (i.e. $d\mu(\mathcal{H}_M) \subset \mathcal{H}$) if and only if $f : M \to N$ is totally geodesic.*

Proof. Using Lemma 4.1.1, we have

$$\begin{aligned}\varphi_{\mathfrak{m}}(JZ^{\mathcal{V}}) &= J_0\varphi_{\mathfrak{n}}(Z^{\mathcal{V}}) \\ &= [\nu_0, \varphi_{\mathfrak{n}}(Z^{\mathcal{V}})] \\ &= -\alpha_{\mathfrak{m}}(\nabla_{d\pi_N(Z)}\xi),\end{aligned}$$

where $\xi(0) = \xi,\ \dot{\xi}(0) = Z$. Since Z is $\pi^\perp$-horizontal and J is an isomorphism, for all $W \in \mathcal{V}_M|_\xi$,

$$\begin{aligned}0 &= h_n(Z^{\mathcal{V}}, W) \\ &= \langle J_0\varphi_{\mathfrak{n}}(Z), J_0\varphi_{\mathfrak{n}}(W)\rangle \\ &= -\left\langle \alpha_{\mathfrak{m}}(\nabla_{d\pi_N(Z)}\xi), \alpha_{\mathfrak{m}}(d\pi_N(JW))\right\rangle \\ &= g(\nabla_{d\pi_N(Z)}\xi, d\pi_N(JW)).\end{aligned}$$

Since $g(\nabla_{d\pi_N(Z)}\xi, \xi) = 0$ and $d\pi_N(J\mathcal{V}_M)+\mathrm{Span}(\xi)$ account for all normal vectors, we observe that

$$A_f(\xi)d\pi_N(Z) = (\nabla_{d\pi_N(Z)}\xi)^\top = \nabla_{d\pi_N(Z)}\xi = d\pi_N(JZ^{\mathcal{V}}).$$

(here we've used the definition of the shape operator (15) to observe that $g(A_f(\xi)X, Y) = g(Y, \nabla_X \xi)$ for $X, Y \in TM,\ \xi \in TM^\perp$). $\square$

Since $UM^\perp$ is Legendrian in UN, it is Lagrangian in $\mathcal{C}$. Hence from Lemma 2.1.10, $h_n(JT(UM^\perp), T(UM^\perp)) = 0$, and so the splitting we shall want to construct our adapted frame for is given by

$$\mu^{-1}\mathcal{C} = \mathcal{H}_M \oplus \mathcal{V}_M \oplus J\mathcal{H}_M \oplus J\mathcal{V}_M. \tag{19}$$

Since $\mathcal{V}_M \subset \mathcal{V} \cong [\mathfrak{n}]_H$, we can note that $J\mathcal{V}_M \subset \mathcal{H}$.

We shall now construct a local adapted orthonormal moving frame for $\mathcal{C}$ about ξ. We choose a local orthonormal frame for a sufficiently small choice of $V \subset UM^\perp$ as

$$\mathcal{H}_M = \mathrm{Span}\{E_j : j = 1, ..., m\}\,; \ \mathcal{V}_M = \mathrm{Span}\{E_\beta : \beta = m+1, ..., n-1\}\,. \tag{20}$$

We then expand this to $V_{\mathcal{C}} \subset \mathcal{C}$ as $E_j, E_\beta, JE_j, JE_\beta$. We shall use the index conventions $i, j \in \{1, ..., m\}$, $\alpha, \beta \in \{m+1, ..., n-1\}$ and $A, B \in \{1, ..., n-1\}$. If we define $W_j = d\pi_N(E_j)$ and $W_\beta = d\pi_N(JE_\beta)$, then $TM = \mathrm{Span}\{W_j\}$ and $TM^\perp = \mathrm{Span}\{W_\beta, \xi\}$. Since $J\mathcal{V}_M \subset \mathcal{H}$, this gives an orthonormal frame for $T_\xi M^\perp$, but not one for TM unless f is totally geodesic.

4.2 The tension field of the Gauss map

Now that we have an adapted frame for the splitting of $\mathcal{C}$, we turn our attention towards the tension field $\tau(\mu)$ of the spherical Gauss map. In particular, we shall be looking to prove the following result.

Theorem 4.2.1. *At a point $\xi \in UM^\perp$ with the adapted frame described in (20) for a neighbourhood $V \subset UM^\perp$ about ξ we have:*

$$h_n(\tau(\mu), JE_i) = -\sum_{j=1}^{m} g\left(\left(\nabla_{d\pi_N(E_i)}\mathrm{I\!I}_f\right)(d\pi_N(E_j), d\pi_N(E_j)), \xi\right),$$

$$h_n(\tau(\mu), JE_\beta) = \sum_{j=1}^{m} g\left(\mathrm{I\!I}_f(d\pi_N(E_j), d\pi_N(E_j)), d\pi_N(JE_\beta)\right).$$

In order to prove this theorem, we shall start by splitting $\tau(\mu)$ according to (19). By Lemma 3.3.3, $\tau(\mu)$ is horizontal with respect to $\pi_{\mathcal{Q}}$ and thus lies in $\mu^{-1}\mathcal{C}$. By noting that $(T(UM)^{\perp})^{\perp} = (\mathcal{H}_M \oplus \mathcal{V}_M)^{\perp}$,

$$\begin{aligned}
\tau(\mu) = \mathrm{tr}_{\mu^* h} \mathrm{I\!I}_\mu &= \sum_{A=1}^{n-1} \mathrm{I\!I}_\mu(E_A, E_A) \\
&= \sum_{A=1}^{n-1} \left(\nabla^\mu_{E_A} d\mu(E_A)\right)^{\perp} \\
&= \sum_{A,B=1}^{n-1} h_n(\nabla^\mu_{E_A} d\mu(E_A), JE_B) JE_B.
\end{aligned}$$

From (18), using the antisymmetry of the Lie bracket to eliminate the last term:

$$\beta_H(\nabla^\mu_{E_A} d\mu(E_A)) = \mathrm{Ad}_\Phi \left(E_A \varphi_{\mathfrak{p}}(E_A) + [\varphi_{\mathfrak{h}}(E_A), \varphi_{\mathfrak{p}}(E_A)]\right).$$

We shall define a set of functions $\tau(\mu)^A : V \to \mathbb{R}$ such that

$$\tau(\mu) = \sum_{i=1}^{m} \tau(\mu)^i JE_i + \sum_{\beta=m+1}^{n-1} \tau(\mu)^\beta JE_\beta.$$

Since $\mathcal{V}_M \subset \mathcal{V}$, $\varphi_{\mathfrak{m}}(E_\beta) = 0$, and so

$$\begin{aligned}
\tau(\mu)^i &= \sum_{j=1}^{m} \langle E_j \varphi_{\mathfrak{p}}(E_j) + [\varphi_{\mathfrak{h}}(E_j), \varphi_{\mathfrak{p}}(E_j)], \varphi_{\mathfrak{p}}(JE_i)\rangle \\
&\qquad + \sum_{\beta=m+1}^{n-1} \langle E_\beta \varphi_{\mathfrak{n}}(E_\beta) + [\varphi_{\mathfrak{h}}(E_\beta), \varphi_{\mathfrak{n}}(E_\beta)], \varphi_{\mathfrak{n}}(JE_j)\rangle, \\
\tau(\mu)^\beta &= \sum_{j=1}^{m} \langle E_j \varphi_{\mathfrak{m}}(E_j) + [\varphi_{\mathfrak{h}}(E_j), \varphi_{\mathfrak{m}}(E_j)], \varphi_{\mathfrak{m}}(JE_\beta)\rangle,
\end{aligned}$$

where we have also made use of the fact that $[\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{k}$. We can use the identities $[\mathfrak{h}, \mathfrak{n}] \subseteq \mathfrak{k}$, $[\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}$ to further decompose $\tau(\mu)^i$ as

$$\begin{aligned}
\tau(\mu)^i_1 &= \sum_{j=1}^{m} \langle E_j \varphi_{\mathfrak{m}}(E_j) + [\varphi_{\mathfrak{h}}(E_j), \varphi_{\mathfrak{m}}(E_j)], \varphi_{\mathfrak{m}}(JE_i)\rangle \\
&\qquad + \sum_{j=1}^{m} \langle E_j \varphi_{\mathfrak{n}}(E_j) + [\varphi_{\mathfrak{h}}(E_j), \varphi_{\mathfrak{n}}(E_j)], \varphi_{\mathfrak{n}}(JE_i)\rangle, \\
\tau(\mu)^i_2 &= \sum_{\beta=m+1}^{n-1} \langle E_\beta \varphi_{\mathfrak{n}}(E_\beta) + [\varphi_{\mathfrak{h}}(E_\beta), \varphi_{\mathfrak{n}}(E_\beta)], \varphi_{\mathfrak{n}}(JE_i)\rangle.
\end{aligned}$$

We can eliminate the second term by considering the following results.

Lemma 4.2.2. *Let $f : M \to N \cong G/K$ be an isometrically immersed submanifold of a CROSS. Let $\tau(U_pM^\perp)$ denote the tension field for the induced immersion $U_pM^\perp \to U_{f(p)}N \cong K/H$, with the normal metric on the sphere K/H. Then for any $Z \in \mathcal{V}$ normal to $U_pM^\perp$,*

$$h_n(\tau(U_pM^\perp), Z) = \sum_{\beta=m+1}^{n-1} \langle E_\beta\varphi_{\mathfrak{n}}(E_\beta) + [\varphi_{\mathfrak{h}}(E_\beta), \varphi_{\mathfrak{n}}(E_\beta)], \varphi_{\mathfrak{n}}(Z)\rangle . \tag{21}$$

Corollary 4.2.3. *When $N = S^n$, $\tau(\mu)^i_2 = 0$.*

Proof. Using the left action of G, we may assume that $f(p)$ is the basepoint $[K]$ of G/K, and so consider $U_pM^\perp \subset K/H$. Since $F([K])K = K$, the restriction of Φ to the fibre must also be K-valued. Since $TK/H \cong [\mathfrak{n}]_H$, we can therefore (using $\Phi : V_p \to K$ as a local frame) calculate the tension field as

$$\sum_{\beta=m+1}^{n-1} \left(\mathrm{Ad}_\Psi \left(E_\beta\varphi_{\mathfrak{n}}(E_\beta) + [\varphi_{\mathfrak{h}}(E_\beta), \varphi_{\mathfrak{n}}(E_\beta)]\right)\right)^\perp ,$$

where the perpendicular is taken within $[\mathfrak{n}]_H$.

In the case where $N = S^n$, $G = SO(n+1)$ and so K/H is the round sphere, as in Example 1.3.3. The fibres $U_pM^\perp$ are themselves equatorial subspheres of the totally geodesic fibres of π_N, and thus totally geodesic. Comparing (21) with $\tau(\mu)^i_2$, we see that when this is the case they both vanish. □

In order to understand $\tau(\mu)^i_1$, we shall want to use the canonical connection D_X defined in §1.3. By comparing it with the Levi-Civita connection for a reductive homogeneous space and using Corollary 1.2.11, we define an operator $D : T(UM^\perp) \times C^\infty(T(UM^\perp), \mathfrak{p}) \to C^\infty(T(UM^\perp), \mathfrak{p})$ such that that its pullback to V takes the form

$$\beta_H(\nabla^{\mathrm{can}}_X Y) = D_X\varphi_{\mathfrak{p}}(Y) = X\varphi_{\mathfrak{p}}(Y) + [\varphi_{\mathfrak{h}}(X), \varphi_{\mathfrak{p}}(Y)]. \tag{22}$$

In terms of D,

$$\tau(\mu)^i_1 = \sum_{j=1}^{m} \left\langle D_{E_j}\varphi_{\mathfrak{m}}(E_j), \varphi_{\mathfrak{m}}(JE_i)\right\rangle + \sum_{j=1}^{m} \left\langle D_{E_j}\varphi_{\mathfrak{n}}(E_j), \varphi_{\mathfrak{n}}(JE_i)\right\rangle .$$

If we consider the decomposition of $\mathfrak{q}$ for S^n given by 11, we can observe that $[\mathfrak{n}, \mathfrak{m}_0] = \mathfrak{s}$. Since $T(UM^\perp)$ lies in the contact distribution, for any $X, Y, Z \in T(UM^\perp)$ we can thus note that

$$\langle [\varphi_{\mathfrak{n}}(X), \varphi_{\mathfrak{m}}(Y)], \varphi_{\mathfrak{m}}(JZ) \rangle = 0. \tag{23}$$

By using Lemma 4.1.2,

$$\varphi_{\mathfrak{m}}(JE_i) = J_0 \varphi_{\mathfrak{n}}(E_i) = J_0 \varphi_{\mathfrak{n}} \left(\overline{A_f(\xi) W_j} \right) = \varphi_{\mathfrak{m}} \left(\overline{A_f(\xi) W_j} \right).$$

Combining these last two observations with (17),

$$\begin{aligned} \left\langle D_{E_j} \varphi_{\mathfrak{m}}(E_j), \varphi_{\mathfrak{m}}(JE_i) \right\rangle &= \langle W_j \alpha_{\mathfrak{m}}(W_j) + [\alpha_{\mathfrak{k}}(W_j), \alpha_{\mathfrak{m}}(W_j)], \alpha_{\mathfrak{m}}(A_f(\xi) W_i) \rangle \\ &= g \left(\nabla^f_{W_j} W_j, A_f(\xi) W_i \right). \end{aligned}$$

Considering (22), since J_0 is Ad_H-invariant $D_X \varphi_{\mathfrak{p}}(JY) = J_0 D_X \varphi_{\mathfrak{p}}(Y)$. As J_0 is an isometry, the second term of $\tau(\mu)^i_1$ becomes

$$\begin{aligned} \left\langle D_{E_j} \varphi_{\mathfrak{n}}(E_j), \varphi_{\mathfrak{n}}(JE_i) \right\rangle &= \left\langle J_0 D_{E_j} \varphi_{\mathfrak{n}}(E_j), J_0 \varphi_{\mathfrak{n}}(JE_i) \right\rangle \\ &= - \left\langle D_{E_j} \varphi_{\mathfrak{m}}(JE_j), \varphi_{\mathfrak{m}}(E_i) \right\rangle \\ &= -g \left(\nabla^f_{W_j} A_f(\xi) W_j, W_i \right). \end{aligned}$$

Since $A_f(\xi)$ is self-adjoint,

$$\begin{aligned} \tau(\mu)^i_1 &= - \sum_{j=1}^m g \left(\nabla^f_{W_j} A_f(\xi) W_j - A_f(\xi) (\nabla^f_{W_j} W_j)^\top, W_i \right) \\ &= - \sum_{j=1}^m g \left(\left(\nabla^f_{W_j} A_f(\xi) \right) W_j, W_i \right), \end{aligned} \tag{24}$$

where ξ moves along the integral curve of W_j.

We would like to relate $\tau(\mu)^i_1$ with the second fundamental form of f. To do so, we differentiate (15). On the left hand side:

$$\begin{aligned} W_j g(A_f(\xi) W_j, W_i) = g &\left((\nabla^f_{W_j} A_f(\xi)) W_j + A_f(\xi) (\nabla^f_{W_j} W_j)^\top, W_i \right) \\ &+ g \left(A_f(\xi) W_j, \nabla^f_{W_j} W_i \right). \end{aligned}$$

On the right hand side:

$$W_j g(\mathrm{II}_f(W_j, W_i), \xi) = g\left(\nabla^f_{W_j} \mathrm{II}_f(W_j, W_i) + \mathrm{II}_f((\nabla^f_{W_j} W_j)^\top, W_i) + \mathrm{II}_f(W_j, (\nabla^f_{W_j} W_i)^\top), \xi\right) + g\left(\mathrm{II}_f(W_j, W_i), \nabla^f_{W_j}\xi\right).$$

As in proof of Lemma 4.1.2, $\nabla^f_{W_j}\xi$ is tangent to M, and so the final term vanishes. Since the shape operator is self-adjoint, by (15) the second and third terms of each side cancel and so

$$\begin{aligned}\tau(\mu)^i_1 &= -\sum_{j=1}^m g\left(\left(\nabla^f_{W_j} A_f(\xi)\right) W_j, W_i\right) = -\sum_{j=1}^m g\left(\left(\nabla^f_{W_j}\mathrm{II}_f\right)(W_j, W_i), \xi\right) \\ &= -\sum_{j=1}^m g\left(\left(\nabla^f_{W_i}\mathrm{II}_f\right)(W_j, W_j), \xi\right). \end{aligned} \tag{25}$$

For the last step, we have used the following result:

Lemma 4.2.4. *[17, Corollary 4.4] Given an isometric immersion $f : M \to (N, g)$, if N is of constant sectional curvature,*

$$(\nabla^f_X \mathrm{II}_f)(Y, Z) = (\nabla^f_Y \mathrm{II}_f)(X, Z).$$

Using (23),we can rewrite $\tau(\mu)^\beta$ as

$$\tau(\mu)^\beta = \sum_{j=1}^m \langle E_j \varphi_{\mathfrak{m}}(E_j) + [\varphi_{\mathfrak{k}}(E_j), \varphi_{\mathfrak{m}}(E_j)], \varphi_{\mathfrak{m}}(JE_\beta)\rangle.$$

From (17) this becomes

$$\begin{aligned}\tau(\mu)^\beta &= \sum_{j=1}^m \langle W_j \alpha_{\mathfrak{m}}(W_j) + [\alpha_{\mathfrak{k}}(W_j), \alpha_{\mathfrak{m}}(W_j)], \alpha_{\mathfrak{m}}(W_\beta)\rangle \\ &= \sum_{j=1}^m g\left(\nabla^f_{W_j}(df(W_j)), W_\beta\right). \end{aligned} \tag{26}$$

We can now combine these observations on the various components of $\tau(\mu)$ to prove Theorem 4.2.1.

Proof of Theorem 4.2.1. By Corollary 4.2.3 and (25), since by definition $W_i = d\pi_N(E_i)$, we can observe

$$\tau(\mu)^i = h_n(\tau(\mu), JE_i) = -\sum_{j=1}^m g\left(\left(\nabla_{d\pi_N(E_i)}\mathrm{II}_f\right)(d\pi_N(E_j), d\pi_N(E_j)), \xi\right).$$

Since $\tau(\mu)$ is horizontal with respect to $\pi_{\mathcal{Q}}$ and $TM^{\perp} = \mathrm{Span}\{W_{\beta}, \xi\}$, (26) gives us

$$\tau(\mu)^{\beta} = h_(\tau(\mu), JE_{\beta}) = \sum_{j=1}^{m} g\left(\mathrm{I\!I}_f\left(d\pi_N(E_j), d\pi_N(E_j)\right), d\pi_N(JE_{\beta})\right).$$

□

While we have succesfully related the tension field of μ (and thus γ) with the second fundamental form of f, we have yet to link it to the tension field of f. This is because, as we see from Lemma 4.1.2, $\pi^{\perp}$ is only a Riemannian submersion when f is totally geodesic. Because of this, the local orthonormal frame $\{E_i\}$ for $UM^{\perp}$ does not necessarily project onto a local orthonormal frame for M with which to take the trace through f^*g. In order to relate them, we shall require an additional condition.

Definition 4.2.5. *Let $\pi : (M, h) \to (N, g)$ be a submersion with horizontal bundle $\mathcal{H}$. We say that π is* horizontally conformal *if there exists $s \in C^{\infty}(M; \mathbb{R})$ such that for all horizontal vectors $X^{\mathcal{H}}, Y^{\mathcal{H}} \in \mathcal{H}$,*

$$h(X^{\mathcal{H}}, Y^{\mathcal{H}}) = s^2 g(d\pi(X^{\mathcal{H}}), d\pi(Y^{\mathcal{H}})).$$

In the case that $\pi^{\perp}$ is horizontally conformal with conformal factor s, the frame $\{E_i\}$ on $V \subset UM^{\perp}$ defines a frame $\{sW_i : W_i = d\pi_N(E_i)\}$ for $U = \pi^{\perp}(V) \subset M$ such that

$$g\left(sW_i, sW_j\right) = h_n(E_i, E_j) = \delta_{ij}.$$

Taking the trace of $\mathrm{I\!I}_f$ with respect to this frame then gives

$$\tau(f) = s^2 \sum_{j=1}^{m} \mathrm{I\!I}_f(d\pi_N(E_j), d\pi_N(E_j)).$$

In order to better understand the significance of this conformal factor, we shall consider the following definition.

Definition 4.2.6. *Let $f : M \to (N, g)$ be an isometric immersion. The* shape form *associated with f is the operator*

$$a_f : TM^\perp \times TM \times TM \to \mathbb{R};\ a_f(\xi)(X, Y) = g(A_f(\xi)X, A_f(\xi)Y).$$

We say f has conformal shape form *when there exists $r \in C^\infty(UM^\perp, \mathbb{R})$ such that $a_f(\xi) = r(\xi)^2 g$ for all $\xi \in UM^\perp$. Since the shape operator is self-adjoint this is equivalent to the condition that $A_f(\xi)^2 = r(\xi)^2 I$ (and so $r(\xi)^2 = \frac{1}{m}\mathrm{tr}(A_f(\xi)^2)$).*

The equivalent condition of $A_f(\xi)^2$ being proportional to the identity is also referred to as *conformal second fundamental form* in [10].

Lemma 4.2.7. *[10, Proposition 4.2] Let $f : M \to (N, g)$ be an isometric immersion. The projection $\pi^\perp : (UM^\perp, \mu^* h_s) \to (M, f^* g)$ is horizontally conformal if and only f has conformal shape form.*

To see how the conformal factors relate, we return to Lemma 4.1.2. If $X_1, X_2 \in \mathcal{H}_M|_\xi$ have projections $Y_i = d\pi_N(X_i)$, then since J is an isometry for h_n:

$$\begin{aligned} h_n(X_1, X_2) &= h_n(\bar{Y}_1, \bar{Y}_2) + h_n(JA_f(\xi)\bar{Y}_1, JA_f(\xi)\bar{Y}_2) \\ &= g(Y_1, Y_2) + g(A_f(\xi)Y_1, A_f(\xi)Y_2). \end{aligned}$$

If $\pi^\perp$ is horizontally conformal with conformal factor s and thus a_f is conformal with conformal factor r:

$$\frac{1}{s^2} g(Y_1, Y_2) = (1 + r^2) g(Y_1, Y_2).$$

By using these conformality conditions, Theorem 4.2.1 gives the following result.

Theorem 4.2.8. *Let $f : M \to (S^n, g)$ be an isometrically immersed submanifold such that $S^n \cong SO(n+1)/K$ and $US^n \cong SO(n+1)/H$ are equipped with the* *normal metric. Let $Z, W \in T_\xi(UM^\perp)$ such that Z is $\pi^\perp$-horizontal and W is vertical. If f has conformal shape form, then*

$$h_n(\tau(\mu), JZ) = -\frac{1}{1+r(\xi)^2} g\left(\nabla^\perp_{d\pi_N(Z)}\tau(f), \xi\right),$$
$$h_n(\tau(\mu), JW) = \frac{1}{1+r(\xi)^2} g\left(\tau(f), d\pi_N(JW)\right),$$

where $r(\xi)^2 = \frac{1}{dim(M)}\mathrm{tr}A_f(\xi)^2$.

When M is a hypersurface, $T_{\pi(\xi)}M^\perp = \mathrm{Span}\{\xi\}$, and so $\mathcal{V}_M = \{0\}$. Using Lemma 3.3.3 and Lemma 3.3.5, we immediately acquire the following corollary.

Corollary 4.2.9. *Let $f : M \to S^n$ have conformal shape form. Then μ is minimal (and γ is minimal Lagrangian) if and only if*

1. *f is minimal, for* $\mathrm{codim}(M) > 1$,
2. *f has constant mean curvature, for* $\mathrm{codim}(M) = 1$.

In order to understand when examples are possible, let us consider the eigenvalues κ_i of the shape operator $A_f(\xi)$. In order for f to have conformal shape form, we must have $\kappa_i^2 = \kappa_j^2$ for all i, m. If we were also to require minimality, then $\sum_{i=1}^m \kappa_i = 0$. Combining these two, f can only be minimal and have conformal shape form if each $A_f(\xi)$ has the same number of positive and negative eigenvalues. Thus for an odd-dimensional submanifold of $\mathrm{codim}(M) > 1$, the only examples occur when f is totally geodesic, and thus each $\kappa_i = 0$.

Example 4.2.10. Every minimal surface $f : M \to S^n$ necessarily has conformal shape form since $A_f(\xi)$ only has two eigenvalues, $\pm\kappa$. Their images $\gamma_f(M)$ thus provide a supply of minimal Lagrangian submanifolds of the oriented Grassmannian

$$\mathcal{Q} \cong Gr(2, n+1) \cong SO(n+1)/SO(n-1) \times S^1.$$

Example 4.2.11. In the case of a hypersurface $f : M \to S^n$ ($n \geq 3$), we can relate our result with Palmer's condition [24] that the classical Gauss map of any isoparametric hypersurface of $S^n \subset \mathbb{R}^{n+1}$ (i.e. one with constant principal curvatures) is minimal. To do this, we first observe that if we consider S^n as a submanifold of $\mathbb{R}^{n+1}$ by the standard embedding, then the geodesic generated by a vector v at p is the great circle given by $S^n \cap Span\{p, v\}$, where we have identified $T_p\mathbb{R}^{n+1}$ with $\mathbb{R}^{n+1}$. For a hypersurface, $\pi^\perp$ identifies the connected component of $UM^\perp$ with M, and so the geodesic Gauss map can thus be identified with the classical Gauss map

$$\hat{\gamma} : M \to \mathrm{Gr}(2, n+1) \cong \mathcal{Q};\ p \mapsto Span\{p, \xi_p\},$$

where ξ_p is the oriented unit normal vector at p. Since $\gamma = \hat{\gamma} \circ \pi^\perp$, by the composition formula for tension fields [9, Proposition 2.20],

$$\tau(\gamma) = d\hat{\gamma}(\tau(\pi^\perp)) + \mathrm{tr}_{g_{\mathcal{Q}}} \nabla d\hat{\gamma}(d\pi^\perp, d\pi^\perp).$$

When f has conformal shape form with conformal factor $r(\xi)^2$, this becomes

$$\tau(\gamma) = d\hat{\gamma}(\tau(\pi^\perp)) + \frac{1}{1 + r(\xi)^2}\tau(\hat{\gamma}).$$

When f has constant mean curvature, by Theorems 3.2.1 and 4.2.8,

$$\tau(\gamma) = \tau(\hat{\gamma}) = 0,$$

and so $\pi^\perp$ is harmonic. By [1, Theorem 5.2], a horizontally conformal submersion into a manifold of dimension ≥ 3 is harmonic with minimal fibres if and only if it is horizontally homethetic. Hence, $r(\xi)$ is constant and so when f has conformal shape form and constant mean curvature, it is isoparametric.

We can in fact reacquire Palmer's formula [24, Proposition 3.4] for the mean curvature of the classical Gauss map of a hypersurface in S^n:

$$g_{\mathcal{Q}}(JH_{\hat{\gamma}}, \cdot) = \sum_{j=1}^{n-1} d(\arctan(\kappa_j)),$$

where κ_j are the principal curvatures of f.

In the case of a hypersurface $\mathcal{V}_M \oplus J\mathcal{V}_M$ is trivial. We can then choose our frame $\{E_i, JE_i : i = 1, .., n-1\}$ for $\mu^{-1}\mathcal{C}$ such that $W_i = d\pi_N(E_i)$ are eigenvectors of the shape operator with eigenvalues κ_i. In this case $E_i = \bar{W}_i - \kappa_i J\bar{W}_i$, and so $||W_i||^2 = \frac{1}{1+\kappa_i^2}$. From (24), we thus see:

$$\begin{aligned} h_n(\tau(\mu), JE_i) &= -\sum_{j=1}^{n-1} g\left((\nabla^f_{W_i} A_f)W_j, W_j\right) \\ &= -\sum_{j=1}^{n-1} g\left(d\kappa_j(E_i)W_j, W_j\right) \\ &= -\sum_{j=1}^{n-1} \frac{1}{1+\kappa_j^2} d\kappa_j(E_i), \end{aligned}$$

and so

$$g_{\mathcal{Q}}(JH_{\hat{\gamma}}, \cdot) = \sum_{j=1}^{n-1} d(\arctan(\kappa_j)).$$

5 Harmonicity of the geodesic Gauss map over $\mathbb{CP}^n$

5.1 Decompositions of the contact structure

We shall now turn our attention to the case of submanifolds of the complex projective space $\mathbb{CP}^n$. In the sphere case the situation was simplified somewhat by the identification of the metrics $h_s = h_n = h_{\mathcal{Q}}$, but that is no longer the case.

To see why, consider the proof of Lemma 2.5.4. The operator $\mathrm{ad}^2_{\nu_o} : \mathfrak{q} \to \mathfrak{q}$ has two eigenspaces for $\mathfrak{g} = \mathfrak{su}(n+1)$. These are the eigenspaces $\mathfrak{q}_1$ and $\mathfrak{q}_2$ with eigenvalues -1 and -4 respectively. As we can see from their decomposition in Lemma 2.5.4 they take the form

$$\mathfrak{q}_2 = \mathrm{Span}\{\mathcal{I}_0\nu_0, \mathrm{ad}_{\nu_0}(\mathcal{I}_0\nu_0)\},\ \ \mathfrak{q}_1 = \mathfrak{q} \cap \mathfrak{q}_2^{\perp},$$

where $\mathcal{I}$ is the standard complex structure on $\mathbb{CP}^n$. We define a corresponding decomposition for the contact structure by

$$\mathcal{C} = \mathcal{C}_1 \oplus \mathcal{C}_2 = \beta_H^{-1}([\mathfrak{q}_1]_H) \oplus \beta_H^{-1}([\mathfrak{q}_2]_H),$$

where

$$\mathcal{C}_2|_\xi = \mathrm{Span}\{\overline{\mathcal{I}\xi}, J\overline{\mathcal{I}\xi}\}.$$

If we let Z_i denote the projection of $Z \in \mathcal{C}$ onto $\mathcal{C}_i$, we define an operator

$$\mathcal{L} : \mathcal{C} \to \mathcal{C};\ Z \mapsto\ Z_1 + \frac{1}{2}Z_2,$$

with a corresponding operator $\mathcal{L}_0$ on $\mathfrak{q}$, then the complex structure J on $\mathcal{C}$ takes the form

$$J_0 = \mathcal{L}_0 \mathrm{ad}_{\nu_0}.$$

Remark 5.1.1. In order to understand the geometric significance of the eigenvalues of $\mathrm{ad}^2_{\nu_0}$, we can consider the sectional curvature of $\mathbb{CP}^n$. Since $\mathbb{CP}^n$ has constant holomorphic sectional curvature, as show in [17, Proposition 7.3], for $S, T, U, V \in TN$:

$$g(R^N(S,T)U,V) = -g(S,U)g(T,V) + g(S,V)g(T,U) - g(S,\mathcal{I}U)g(T,\mathcal{I}V) \\ + g(S,\mathcal{I}V)g(T,\mathcal{I}U) - 2g(S,\mathcal{I}T)g(U,\mathcal{I}V). \tag{27}$$

For orthonormal $U, V \in T_pN$, the sectional curvature of $\mathrm{Sp}\{U,V\}$ is thus given by

$$k(U,V) = \frac{g(R^N(U,V)V,U)}{||U||^2||V||^2 - g(U,V)^2} = 1 + 3g(U,\mathcal{I}V)^2.$$

The extremal values of k are thus given by

- $k = 1$, when $\mathrm{Sp}\{U,V\} \subseteq T_pN$ is an isotropic subplane,
- $k = 4$. when $\mathrm{Sp}\{U,V\} \subseteq T_pN$ is a holomorphic subplane.

By Lemma 1.3.1, for $X \in U_pM,\ \xi \in U_pM^\perp$ such that $\alpha_{\mathfrak{m}}(\xi) = \nu_0$:

$$\begin{aligned}\left\langle \mathrm{ad}^2_{\nu_0}\alpha_{\mathfrak{m}}(X), \alpha_{\mathfrak{m}}(X)\right\rangle &= -g(R^N(X,\xi)\xi, X) \\ &= -1 - g(X,\mathcal{I}\xi).\end{aligned}$$

The eigenvalues of $\mathrm{ad}^2_{\nu_0}$ thus correspond to the extreme values of the sectional curvature.

From Definition 2.5.8 we can see that when restricted to $\mathcal{C}$,

$$h_s(X,Y) = h_n(X^{\mathcal{H}}, Y^{\mathcal{H}}) - h_n(\mathcal{L}^{-2}X^{\mathcal{V}}, Y^{\mathcal{V}}).$$

(where we have used the fact that $\mathcal{L}$ is clearly self-adjoint, since $h_n(\mathcal{C}_1, \mathcal{C}_2) = 0$ and it is simply a scaling within each $\mathcal{C}_i$). We thus define an operator

$$B : TUN \to TUN;\ X \mapsto X^{\mathcal{H}} - \mathcal{L}^{-2}X^{\mathcal{V}}$$

such that

$$h_s(X,Y) = h_n(BX, Y).$$

From (10), for vectors $X, Y \in \mathcal{C}$:

$$\begin{aligned} h_{\mathcal{Q}}(X,Y) &= \langle \nu_0, [\varphi_{\mathfrak{p}}(X), \varphi_{\mathfrak{p}}(JY)]\rangle \\ &= \langle \nu_0, [\varphi_{\mathfrak{p}}(X), [\nu_0, \varphi_{\mathfrak{p}}(\mathcal{L}Y)]]\rangle \\ &= -\left\langle \mathrm{ad}^2_{\nu_0}\varphi_{\mathfrak{p}}(X), \varphi_{\mathfrak{p}}(\mathcal{L}Y)\right\rangle \\ &= \left\langle \varphi_{\mathfrak{p}}(\mathcal{L}^{-2}X), \varphi_{\mathfrak{p}}(\mathcal{L}Y)\right\rangle \\ &= h_n(\mathcal{L}^{-2}X, \mathcal{L}Y) \\ &= h_n(\mathcal{L}^{-1}X, Y). \end{aligned}$$

Hence,

$$h_n(X,Y) = h_{\mathcal{Q}}(\mathcal{L}X, Y) = h_s(B^{-1}X, Y). \tag{28}$$

By definition J is an isometry for $h_{\mathcal{Q}}$. If we consider h_n:

$$h_n(JX, JY) = h_{\mathcal{Q}}(JX_1, JY_2) + \frac{1}{2}h_{\mathcal{Q}}(JX_2, JY_2).$$

Since $\mathcal{C}_j$ are J-invariant, we observe that J is also an isometry for h_n

All three metrics present some issues. As we can see, $h_n \neq h_{\mathcal{Q}}$, and by the uniqueness of the Kähler-Einstein structure on $\mathcal{Q}$, $h_n|_{\mathcal{Q}}$ is not Kähler-Einstein. With regards to $h_{\mathcal{Q}}$, since $\mathcal{C}_2 \cap \mathcal{H} \neq \{0\}$ the above equation implies π_N is not a Riemannian submersion with respect to $h_{\mathcal{Q}}$. By Lemma 2.5.9, h_s doesn't descend to a metric on $\mathcal{Q}$.

We shall thus consider separately the cases where UN is equipped with the metrics h_n and $h_{\mathcal{Q}}$. In particular, we shall be studying two types of submanifolds of $\mathbb{CP}^n$, holomorphic and coisotropic submanifolds. As shown in Lemma 2.1.10, these are the submanifolds such that $\mathcal{I}TM = TM$ and $\mathcal{I}TM^{\perp} \subseteq TM$, respectively. These submanifolds have the useful property that even though π_N is not Riemannian, its horizontal bundles align with the horizontal bundles for h_n. We can thus still lift vectors from TN to $\mathcal{H}$, although the lift won't be length preserving.

Lemma 5.1.2. *Let $\mathcal{H}$ and $\mathcal{H}_M$ be the horizontal distributions with respect to h_n for π_N and $\pi^\perp$ respectively. Then $h_{\mathcal{Q}}(\mathcal{H}, \mathcal{V}) = 0$ and when $f : M \to \mathbb{CP}^n$ is a holomorphic or coisotropic submanifold, $h_{\mathcal{Q}}(d\mu(\mathcal{H}_M), d\mu(\mathcal{V}_M)) = 0$.*

Proof. For $\mathcal{H}$, simply note that the action of $\mathcal{L}$ on $\mathcal{V}$ is just rescaling the one-dimensional subspace V_2, and thus $\mathcal{L}\mathcal{V} = \mathcal{V}$. For $\mathcal{H}_M$, we note that the holomorphic and coisotropic cases imply that $(\mathcal{C}_2 \cap T(UM^\perp))$ is contained within $\mathcal{V}_M$ and $\mathcal{H}_M$ respectively. Hence, $\mathcal{L}\mathcal{H}_M = \mathcal{H}_M$ for the holomorphic case and $\mathcal{L}\mathcal{V}_M = \mathcal{V}_M$ in the coisotropic case. $\square$

Because of this, we can adapt Lemma 4.1.2 to provide a description of the horizontal decomposition of $UM^\perp$ with respect to both metrics.

Lemma 5.1.3. *Let $f : M \to \mathbb{CP}^n$ be an isometrically immersed submanifold. Let $Z \in T_\xi(UM^\perp)$ be a $\pi^\perp$-horizontal vector with respect to h_n. Then Z splits with respect to π_N as*

$$Z^{\mathcal{H}} = \overline{d(\pi_N \cdot \mu)(Z)};$$
$$Z^{\mathcal{V}} = -J\overline{A_f(\xi) d(\pi_N \cdot \mu)(\mathcal{L}Z)}.$$

Corollary 5.1.4. *When M is a holomorphic submanifold, $Z \in \mathcal{H}_M$ is $\pi^\perp$-horizontal with respect to both h_s and $h_{\mathcal{Q}}$ and the decomposition is given by*

$$Z^{\mathcal{H}} = \overline{d(\pi_N \cdot \mu)(Z)}$$
$$Z^{\mathcal{V}} = -J\overline{A_f(\xi) d(\pi_N \cdot \mu)(Z)}$$

and the restrictions of h_n and $h_{\mathcal{Q}}$ to $\mathcal{H}_M$ agree with the restriction of the Sasaki metric.

Corollary 5.1.5. *When M is a coisotropic submanifold, $Z \in \mathcal{H}_M$ is $\pi^\perp$-horizontal for both h_n and $h_{\mathcal{Q}}$, and the decomposition is given by*

$$Z^{\mathcal{H}} = \overline{d(\pi_N \cdot \mu)(Z)}$$
$$Z^{\mathcal{V}} = -J\overline{B_f(\xi) d(\pi_N \cdot \mu)(Z)},$$

where B_f is the operator

$$B_f : UM^\perp \times TM \to TM;$$
$$(\xi, X) \mapsto \left(A_f(\xi)X - \frac{1}{2}g(X, \mathcal{I}\xi)A_f(\xi)\mathcal{I}\xi \right),$$

which we shall refer to as the adapted shape operator*.*

Proof. The proof of the Lemma is identical to the proof of Lemma 4.1.2 using $J_0\varphi_{\mathfrak{n}}Z = \mathrm{ad}_{\nu_0}(\mathcal{L}Z)$, where we have equipped neighbourhoods $V \subset UM^\perp, U \subset M$ with local frames Φ, F as in (16).

For the first corollary we consider the splitting

$$\mu^{-1}\mathcal{C} = \mathcal{H}_M \oplus \mathcal{V}_M \oplus J\mathcal{H}_M \oplus J\mathcal{V}_M.$$

To see that this is an orthogonal splitting for h_n as well as $h_{\mathcal{Q}}$, we note that J preserves $\mathcal{C}_2$ and is thus an isometry for both metrics. For the holomorphic case $\mathcal{I}\xi \in TM^\perp$ and so $\overline{\mathcal{I}\xi} \in J\mathcal{V}_M$. Hence, $\mathcal{C}_2 \subseteq \mathcal{V}_M \oplus J\mathcal{V}_M$ and $\mathcal{L}|_{\mathcal{H}_M} = \mathcal{B}|_{\mathcal{H}_M} = \mathrm{id}$.

For the second corollary, when M is coisotropic $\mathcal{I}TM^\perp \subset TM$, and so $\mathcal{C}_2 \subseteq \mathcal{H}_M \oplus J\mathcal{H}_M$. Since $\mathcal{I}\xi \in TM$, $A_f(\xi)\mathcal{I}\xi$ is well defined and $\mathcal{C}_2 \cap \mathcal{H}_M = \mathrm{Span}\{\mathcal{I}\xi\}$. □

Another reason for using holomorphic and coisotropic submanifolds is when we consider the fibres of $UM^\perp$. When working with spheres we were able to eliminate the π_N-vertical components of the tension field, indexed as $\tau(\mu)_2^i$, by observing that the spherical Gauss map had minimal fibres, a property of the Sasaki metric. While the differences between the metrics when working with complex projective spaces mean this isn't usually the case, for holomorphic and coisotropic submanifolds we can prove the following result.

Proposition 5.1.6. *Suppose $f : M \to \mathbb{CP}^n$ is an immersion which is either holomorphic or coisotropic. Then the restriction of μ to the fibres of $UM^\perp$ is minimal with respect to h_n and $h_{\mathcal{Q}}$.*

To prove this, we need a few preliminary results. In the sphere case, the fibres being minimal was a consequence of the use of the Sasaki metric h_s. If we define

$$\mathfrak{um} = \{\xi \in \mathfrak{m} : \ \langle \xi, \xi \rangle = 1\},$$

then $\mathfrak{um} \cong K/H$. The round metric on $\mathfrak{um}$ is thus the metric induced by the inclusion $\mathfrak{um} \to \mathfrak{m}$, and so by Definition 2.5.8 it corresponds to the restriction of the Sasaki metric.

To relate the tension fields of μ with respect to the various metrics, we'll want to consider their Levi-Civita connections. If we define for metrics h_a, h_b the tensor $A^a_b = \nabla^a - \nabla^b$, then by (5) and (28):

$$h_s(A^n_s(X,Y),Z) = -\frac{1}{2}\left\langle [\beta_H(X), \beta_H(BY)] + [\beta_H(Y), \beta_H(BX)], \beta_H(Z) \right\rangle, \quad (29)$$

$$h_{\mathcal{Q}}(A^n_{\mathcal{Q}}(X,Y),Z) = -\frac{1}{2}\left\langle [\beta_H(X), \beta_H(\mathcal{L}^{-1}Y)] + [\beta_H(Y), \beta_H(\mathcal{L}^{-1}X)], \beta_H(Z) \right\rangle, \quad (30)$$

where $\mathcal{L}^{-1} : \mathcal{C} \to \mathcal{C};\ Z \mapsto Z_1 + 2Z_2$.

Lemma 5.1.7. *Let $\mathfrak{v} \subset \mathfrak{m}$ be a proper subspace and $\Sigma = \mathfrak{v} \cap \mathfrak{um}$ the corresponding unit subsphere. Suppose that the tangent bundle $T\Sigma$ is*

1. *a B-invariant subbundle of $T\mathfrak{um}$. Then Σ is a minimal submanifold for the normal metric on $\mathfrak{um}$,*
2. *a $(\mathcal{L} \cdot B)$-invariant subbundle of $T\mathfrak{um}$. Then Σ is a minimal submanifold for $h_{\mathcal{Q}}$ on $\mathfrak{um}$.*

Proof. Since B (corresponding to the Ad_H-invariant operator given by $\mathrm{id}|_{\mathfrak{m}}$ and $-\mathrm{ad}^2_{\nu_0}|_{\mathfrak{n}}$) is self-adjoint for both h_n and h_s, then $T\Sigma$ has the same normal bundle for both metrics and $B(T\Sigma^{\perp}) = T\Sigma^{\perp}$. We now construct an adapted orthonormal frame $W_1, ..., W_{m-1}$ about $\nu \in \Sigma$ with respect to h_n such that $W_1, ..., W_k \in T\Sigma$ and each W_i is an eigenvector of B with eigenvalue λ_i.

Since $\tau_n(\Sigma) \in T\Sigma^\perp$ with respect to both metrics, we only need consider $W_\alpha \in T\Sigma^\perp$:

$$\begin{aligned}
h_s(\tau_n(\Sigma), W_\alpha) &= h_s\left(\left(\sum_{i=1}^k \nabla^n_{W_i} W_i\right)^\perp, W_\alpha\right) \\
&= \sum_{i=1}^k h_s\left(\nabla^s_{W_i} W_i - A^s_n(W_i, W_i), W_\alpha\right) \\
&= h_s(\tau_s(\Sigma), W_\alpha) - \sum_{i=1}^k h_s\left(A^s_n(W_i, W_i), W_\alpha\right). \qquad (31)
\end{aligned}$$

Since h_s is the round metric for $\mathfrak{um}$, the proper subsphere Σ is totally geodesic with respect to h_s. Hence, from (30):

$$\begin{aligned}
h_s(\tau_n(\Sigma), W_\alpha) &= \frac{1}{2}\sum_{i=1}^k \langle [\beta_H(W_i), \beta_H(BW_i)] + [\beta_H(W_i), \beta_H(BW_i)], \beta_H(W_\alpha)\rangle \\
&= -\sum_{i=1}^k \lambda_i \langle [\beta(W_i), \beta(W_i)], \beta_H(W_\alpha)\rangle = 0.
\end{aligned}$$

Hence, $\Sigma \subset \mathfrak{um}$ is minimal with respect to h_n.

The proof of the second case is almost identical, except substituting $(\mathcal{L}\cdot B)$ for B, $h_\mathcal{Q}$ for h_n and $A^s_\mathcal{Q}$ for A^s_n. It diverges however when we reach (31). We now note that $B|_\mathcal{V} = \mathcal{L}^{-2}|_\mathcal{V}$, and thus $\beta_H(\mathcal{L}^{-1}W_i) = (\mathcal{L}\cdot B)\beta_H(W_i) = \lambda_i\beta_H(W_i)$, and so

$$\begin{aligned}
h_s(\tau_\mathcal{Q}(\Sigma), W_\alpha) &= -\sum_{i=1}^k h_s((A^s_n + A^n_\mathcal{Q})(W_i, W_i), W_\alpha) \\
&= -\sum_{i=1}^k \left\langle [\beta_H(W_i), \beta_H(\mathcal{L}^{-1}W_i)] - [\beta_H(W_i), \beta_H(BW_i)], \beta_H(W_\alpha)\right\rangle \\
&= -\sum_{i=1}^k \lambda_i \langle [\beta_H(W_i), \beta_H(W_i)] - [\beta_H(W_i), \beta_H(W_i)], \beta_H(W_\alpha)\rangle = 0.
\end{aligned}$$

Hence, $\Sigma \subset \mathfrak{um}$ is minimal with respect to $h_\mathcal{Q}$. □

Proof of Proposition 5.1.6. In order to prove Proposition 5.1.6, we need to consider the eigenspaces of B and $\mathcal{L}\cdot B$. While they differ on $\mathcal{H}$, $B_\mathcal{V} = \mathcal{L}^{-2}|_\mathcal{V}$, and so they both have eigenspaces $V_1 = \mathcal{C}_1 \cap \mathcal{V}$ and $V_2 = \mathcal{C}_2 \cap \mathcal{V}$. Since $\beta_H(V_2) = \mathrm{Ad}_H\left(\mathrm{Span}\{\mathrm{ad}_{\nu_0}\mathcal{I}_0\nu_0\}\right)$ is one-dimensional, given a proper subspace

$\mathfrak{v} \subset \mathfrak{m}$ as before, then for $T\Sigma$ to be B or $(\mathcal{L} \cdot B)$-invariant, either $\beta_H(V_2) \cap T\Sigma = \{0\}$ or $\beta_H(V_2) \subseteq T\Sigma$.

If we again consider the inclusion (12) $\nu : UN \to TN$, then $\beta_K(d\nu_0(V_2)) = \mathrm{Ad}_K(\mathrm{Span}\{\mathcal{I}_0\nu_0\})$. Since $\mathrm{Ad}_K(\mathrm{Span}\{\nu_0\})$ is the normal bundle of $\mathfrak{um} \subset \mathfrak{m}$, $V_2^\perp = V_1$ corresponds to the standard contact structure on $S^{2n-1} \subset \mathbb{C}^n \cong \mathfrak{m}$, such that $V_2 = S^{2n-1} \cap \mathcal{I}S^{2n-1}$, given in [18, Example 3.47]. Hence, by Lemma 2.1.6, when $T\Sigma \subset \beta_H(V_1)$, $\mathfrak{v}$ is isotropic with respect to the symplectic structure $\omega^N = -g(\mathcal{I}\cdot, \cdot)$. In the case that $\mathfrak{v} \cong T_pM^\perp$, M is therefore coisotropic.

Given a one-dimensional subspace $l \subset \mathfrak{v}$, then $l \cap \mathfrak{um} = \pm\mathrm{Ad}_k(\nu_0)$, for some $k \in K$. In the case that $\beta_H(V_2) \subseteq T\Sigma$, $\mathrm{Ad}_k(\mathcal{I}_0\nu_0) \in T\Sigma \subset T\mathfrak{v}$, and so $\mathfrak{v}$ is $\mathcal{I}$-invariant. Hence, when $\mathfrak{v} \cong T_pM^\perp$, $\mathcal{I}T_pM = T_pM$. We have thus proved Proposition 5.1.6. $\square$

5.2 Holomorphic submanifolds

5.2.1 The normal metric

The first case we shall consider is that of a holomorphic submanifold $f : M \to \mathbb{CP}^n$ when UN is equipped with the normal metric h_n.

Theorem 5.2.1. *Let $f : M \to \mathbb{CP}^n$ be a holomorphic submanifold with conformal shape form. Then its spherical and geodesic Gauss maps are minimal with respect to the normal metric.*

As was the case with S^n, in order to prove this theorem we shall want to construct orthonormal adapted local frames with respect to each metric for the decomposition $\mu^{-1}\mathcal{C} = \mathcal{H}_M \oplus \mathcal{V}_M \oplus J\mathcal{H}_M \oplus J\mathcal{V}_M$. In order to ease calculations, we shall make some additional considerations in the construction of this frame, the first being that the frame projects onto eigenvectors of the shape operator. We would also like the frame to respect $\mathcal{I}$, the standard complex structure on

$\mathbb{CP}^n$.

To ensure that our frame can obey both of these conditions simultaneously, we need to verify that the shape operator commutes with the complex structure. If we first note that $\mathbb{CP}^n$ is a Kähler manifold, then $\mathcal{I}$ is a parallel isometry. If we also note that II_f is symmetric, then for any $X, Y \in T_\xi M$:

$$\begin{aligned} g\left(A_f(\xi)(\mathcal{I}X), Y\right) &= g\left(\mathrm{II}(\mathcal{I}X, Y), \xi\right) \\ &= g\left(\nabla^f_Y(df(\mathcal{I}X)) - df(\nabla^M_X Y), \xi\right). \end{aligned}$$

The second term vanishes, and since $\mathcal{I}$ is parallel it passes through the derivative. Since $\mathcal{I}$ is an isometry, we then calculate

$$\begin{aligned} g\left(A_f(\xi)(\mathcal{I}X), Y\right) &= g\left(\mathcal{I}\nabla^f_Y df(X), \xi\right) \\ &= -g\left(\mathrm{II}(X, Y), \mathcal{I}\xi\right) \\ &= -g\left(\mathrm{II}(Y, X), \mathcal{I}\xi\right) \end{aligned}$$

Retracing the calculations with X and Y now reversed, we thus obtain

$$g\left(A_f(\xi)(\mathcal{I}X), Y\right) = g\left(A_f(\xi)X, \mathcal{I}Y\right).$$

Hence, when V is an eigenvector of $A_f(\xi)$ with eigenvalue κ,

$$g(A_f(\xi)(\mathcal{I}V), Y) = -\kappa g(\mathcal{I}V, Y), \tag{32}$$

and so $\mathcal{I}V$ is itself an eigenvector with eigenvalue $-\kappa$.

Remark 5.2.2. An immediate consequence of this is the well known result that all holomorphic submanifolds of $\mathbb{CP}^n$ are minimal, as the paired eigenvalues eliminate each other in the trace of $A_f(\xi)$.

By (32), we can choose a Hermitian frame of eigenvectors $V_1, ..., V_m \in T_{\pi(\xi)}M$of the shape operator $A_f(\xi)$ with corresponding eigenvalues $\kappa_j(\xi)$ such that

$$\mathcal{I}V_{2i-1} = V_{2i}, \quad \kappa_{2i-1} = -\kappa_{2i} \text{ for } i = 1, ..., \frac{m}{2}.$$

We define

$$\tilde{V}_j = \bar{V}_j - \kappa_j J \bar{V}_j,$$

where $\bar{V}_j$ is the horizontal lift with respect to $d\pi_N$. By Corollary 5.1.4, $\tilde{V}_j \in \mathcal{H}_M$. Since J is an isometry and π_N is a Riemannian submersion with respect to h_n:

$$||\tilde{V}_j|| = ||\bar{V}_j - \kappa_j J\bar{V}_j|| = \sqrt{1+\kappa_j^2}.$$

Let $U \subset M,\ V \subset UM^\perp$ be a pair of open subsets such that $\pi^\perp(V) = U$. We can thus define a local orthonormal frame for $\mathcal{H}_M|_V$ as

$$\left\{ E_j :\ E_j|_\xi = \frac{\tilde{V}_j(\xi)}{\sqrt{1+\kappa_j(\xi)^2}}\ j = 1, ..., m \right\}, \tag{33}$$

for a smooth choice of $\tilde{V}_j : V \subset UM^\perp \to TU \subset TM$. We again define $W_j = d\pi_N(E_j)$. By (32), $||\mathcal{I}W_i|| = (1+(-\kappa_i)^2)^{-\frac{1}{2}} = ||W_i||$.

For $\mathcal{V}_M$ we simply choose an h_n-orthonormal frame $\{E_{m+1}, ..., E_{n-1}\}$ for $V \subset UM^\perp$ with corresponding $W_\beta = d\pi_N(JE_\beta)$ such that at $\pi_N(\xi)$:

- $\mathcal{I}W_\beta \in \{\pm W_{m+1}, ..., \pm W_{n-2},\}$, for $\beta \in \{m+1, ..., n-2\}$,
- $\mathcal{I}W_{n-1} = -\xi$.

and Span $\{W_\beta, \xi : \beta = m+1, ..., n-1\} = T_\xi M^\perp$. To see that we can do this, we note that $\mathcal{I}$ and J are isometries for g and h_n respectively and $d\pi_N$ is a Riemannian submersion for h_n.

By the same argument as in §4.2, we again have an orthonormal frame such that

$$\tau_n(\mu) = \sum_i \tau_n(\mu)^i JE_i + \sum_\beta \tau_n(\mu)^\beta JE_\beta, \tag{34}$$

where $\tau_n(\mu)^i = \tau_n(\mu)^i_1 + \tau_n(\mu)^i_2$ with:

$$\begin{aligned} \tau_n(\mu)^i_1 = &\left\langle \sum_j E_j\phi_{\mathfrak{m}}(E_j) + [\varphi_{\mathfrak{h}}(E_j), \varphi_{\mathfrak{m}}(E_j)], \varphi_{\mathfrak{m}}(JE_i) \right\rangle \\ &+ \left\langle \sum_j E_j\varphi_{\mathfrak{n}}(E_j) + [\varphi_{\mathfrak{h}}(E_j), \varphi_{\mathfrak{n}}(E_j)], \varphi_{\mathfrak{n}}(JE_i) \right\rangle, \end{aligned} \tag{35}$$

$$\tau_n(\mu)^i_2 = \left\langle \sum_\beta E_\beta\varphi_{\mathfrak{n}}(E_\beta) + [\varphi_{\mathfrak{h}}(E_\beta), \varphi_{\mathfrak{n}}(E_\beta)], \varphi_{\mathfrak{n}}(JE_i) \right\rangle \tag{36}$$

and

$$\tau_n(\mu)^\beta = \left\langle \sum_j E_j\varphi_{\mathfrak{m}}(E_j) + \left[\varphi_{\mathfrak{h}}(E_j), \varphi_{\mathfrak{m}}(E_j)\right], \varphi_{\mathfrak{m}}(JE_\beta) \right\rangle. \tag{37}$$

Now we can examine each component.

Proposition 5.2.3. *When $f : M \to N = \mathbb{CP}^n$ is holomorphic, $\tau_n(\mu)_2^i = 0$.*

Proof. By Lemma 4.2.2, we see

$$\tau_n(\mu)_2^i = h_n(\tau_n(U_pM^\perp), JE_i).$$

Unlike with spheres, the restriction of μ to the fibres of $UM^\perp$ is not necessarily minimal with respect to the normal metric. However, as we saw in Proposition 5.1.6, they are minimal for holomorphic submanifolds. $\square$

Turning to $\tau(\mu)_1^i$, as in the sphere case, with respect to the canonical connection:

$$\tau(\mu)_1^i = \sum_{j=1}^m \left\langle D_{E_j}\varphi_{\mathfrak{m}}(E_j), \varphi_{\mathfrak{m}}(JE_i)\right\rangle + \sum_{j=1}^m \left\langle D_{E_j}\varphi_{\mathfrak{n}}(E_j), \varphi_{\mathfrak{n}}(JE_i)\right\rangle.$$

For complex projective space, $[\mathfrak{n}, \mathfrak{m}_0] \cap \mathfrak{q} \neq \{0\}$, and so in order to relate the first term to ∇^f, we must utilise the following result.

Lemma 5.2.4. *Let $f : M \to \mathbb{CP}^n$ be a holomorphic submanifold. For a local frame E_i for $\mathcal{H}_M$ as described above,*

$$\langle[\varphi_{\mathfrak{n}}(E_j), \varphi_{\mathfrak{m}}(E_j)], \varphi_{\mathfrak{m}}(\cdot)\rangle = \frac{\kappa_j}{\sqrt{1+\kappa_j^2}} g(\xi, d\pi_N(\cdot)).$$

Proof. By Corollary 5.1.4 and Lemma 1.3.1,

$$\begin{aligned}\langle[\varphi_{\mathfrak{n}}(E_j), \varphi_{\mathfrak{m}}(E_j)], \varphi_{\mathfrak{n}}(\cdot)\rangle &= -\left\langle \left[[\nu_0, \varphi_{\mathfrak{m}}(J\overline{A_f(\xi)W_j})], \varphi_{\mathfrak{m}}(E_j)\right], \varphi_{\mathfrak{n}}(\cdot)\right\rangle \\ &= g\left(R^N(\xi, \kappa_j W_j)W_j, d\pi_N(\cdot)\right).\end{aligned}$$

As in Remark 5.1.1, since $\mathbb{CP}^n$ has constant holomorphic sectional curvature, for $S, T, U, V \in TN$:

$$g(R^N(S,T)U,V) = -g(S,U)g(T,V) + g(S,V)g(T,U) - g(S,\mathcal{I}U)g(T,\mathcal{I}V) \\ + g(S,\mathcal{I}V)g(T,\mathcal{I}U) - 2g(S,\mathcal{I}T)g(U,\mathcal{I}V).$$

Since M is holomorphic,

$$g(\xi, W_j) = g(\xi, \mathcal{I}W_j) = g(W_j, \mathcal{I}W_j) = 0, \tag{38}$$

and so (for $Z \in T_pN$) this reduces to

$$\kappa_j g(R^N(\xi, W_j)W_j, Z) = \kappa_j g(\xi, Z) g(W_j, W_j). \tag{39}$$

By (33),

$$g(W,W) = \frac{1}{\sqrt{1+\kappa_j^2}}.$$

$\square$

With this in mind, following the same argument used to acquire (25), we find

$$\tau_n(\mu)_1^i = -\sum_{j=1}^m g\left(\left(\nabla^f_{W_j} A_f(\xi)\right) W_j, W_i\right) = -\sum_{j=1}^m g\left(\left(\nabla^f_{W_j} \mathrm{I\!I}_f\right)(W_j, W_i), \xi\right).$$

Unlike with S^n, $\nabla \mathrm{I\!I}_f$ is not totally symmetric for $\mathbb{CP}^N$, but we can still rearrange the arguments on the right hand side. The Codazzi equation [17, Proposition 4.3] tells us

$$\nabla^f_{W_j} \mathrm{I\!I}_f(W_j, W_i) = \left(R^N(W_j, W_i)W_j\right)^\perp + \nabla^f_{W_i} \mathrm{I\!I}_f(W_j, W_j).$$

If we consider (38) we can see that when M is holomorphic, for any $Z \in T_pM^\perp$:

$$g(R^N(W_j, W_i)W_j, Z) = 0,$$

since every term of (27) vanishes. Hence,

$$\tau_n(\mu)_1^i = \sum_{j=1}^m g\left(\left(\nabla^f_{W_i} \mathrm{I\!I}_f\right)(W_j, W_j), \xi\right). \tag{40}$$

Now we turn to $\tau_n(\mu)^\beta$. By Lemma 5.2.4,

$$\langle [\varphi_{\mathfrak{n}}(E_j), \varphi_{\mathfrak{m}}(E_j)], \varphi_{\mathfrak{m}}(JE_\beta)\rangle = 0,$$

and so, as in the sphere case,

$$\tau_n(\mu)^\beta = \sum_{j=1}^{m} g\left(\nabla^f_{W_j} df(W_j), W_\beta\right). \tag{41}$$

We now have the necessary preparations to prove Theorem 5.2.1

Proof of Theorem 5.2.1. By Corollary 5.1.4, $h_n|_{\mathcal{H}_M} = h_s|_{\mathcal{H}_M}$ and so Lemma 4.2.7 still applies, i.e. $\pi^\perp$ is h_n-horizontally conformal if and only if f has conformal shape form. As stated in Remark 5.2.2, if f then it is holomorphic it is necessarily minimal. When f is both minimal and has conformal shape form $A_f(\xi)$ has only two eigenvalues, $\pm\kappa$. From (40), (41) and Proposition 5.2.3, we thus observe that when f has conformal shape form:

$$\begin{aligned}
\tau_n(\mu)^i &= -\frac{1}{1+\kappa^2} g\left(\nabla_{W_i}\tau(f), \xi\right) = 0 \\
\tau_n(\mu)^\beta &= \frac{1}{1+\kappa^2} g\left(\tau(f), W_\beta\right) = 0.
\end{aligned}$$

The statement for the geodesic Gauss map then follows from Lemma 3.3.3.

□

5.2.2 The lift of the Kähler-Einstein metric

We shall now consider what happens if we instead equip the unit tangent bundle with h_Q. We can use the local frame from the preceding section to easily construct an h_Q-orthonormal frame:

$$F_A := \begin{cases} E_A, & A \neq n-1 \\ \frac{1}{\sqrt{2}} E_{n-1}, & A = n-1. \end{cases}$$

Using this frame we can adapt Theorem 5.2.1 for h_Q.

Theorem 5.2.5. *Let $f : M \to \mathbb{CP}^n$ be a holomorphic submanifold with conformal shape form. Then its spherical and geodesic Gauss maps are minimal for $h_{\mathcal{Q}}$ and minimal Lagrangian for $g_{\mathcal{Q}}$ respectively.*

Proof. To prove this result, we will want to compare $\tau_{\mathcal{Q}}(\mu)$ and $\tau_n(\mu)$. To do this we use the tensor $A^n_{\mathcal{Q}} = \nabla^n - \nabla^{\mathcal{Q}}$. Using (30), for $X, Z \in TUN$:

$$h_{\mathcal{Q}}(A^n_{\mathcal{Q}}(X,X),Z) = \left\langle [\beta_H(\mathcal{L}^{-1}X), \beta_H(X)], \beta_H(Z)\right\rangle.$$

Since E_A (and thus F_A) are eigenvectors of $\mathcal{L}^{-1}$, $A^n_{\mathcal{Q}}(F_A, F_A) = 0$. Hence,

$$\begin{aligned} h_{\mathcal{Q}}(\tau_{\mathcal{Q}}(\mu), JF_B) &= \sum_{A=1}^{n-1} h_{\mathcal{Q}}\left(\nabla^{\mathcal{Q}}_{F_A} F_A, JF_B\right) \\ &= \sum_{A=1}^{n-1} h_{\mathcal{Q}}\left(\nabla^n_{F_A} F_A - A^n_{\mathcal{Q}}(F_A, F_A), JF_B\right) \\ &= h_{\mathcal{Q}}\left(\tau_n(\mu) - \frac{1}{2}\nabla^n_{E_{n-1}} E_{n-1}, JF_B\right) \\ &= h_n\left(\tau_n(\mu) - \frac{1}{2}\nabla^n_{E_{n-1}} E_{n-1}, \mathcal{L}JF_B\right). \end{aligned}$$

When f is holomorphic with conformal shape form, by Theorem 5.2.1, $\tau_n(\mu) = 0$.

In order to eliminate the remaining term, we now use $A^n_s = \nabla^n - \nabla^s$. By (29), for $Z \in TUN$:

$$h_s\left(A^n_s(E_{n-1}, E_{n-1}), Z\right) = \langle [\beta_H(BE_{n-1}), \beta_H(E_{n-1})], \beta_H(Z)\rangle.$$

Since E_{n-1} lies in $\mathcal{V}_M \subset \mathcal{V}$ and it is an eigenvector of $\mathcal{L}$,

$$BE_{n-1} = E^{\mathcal{H}}_{n-1} - \mathcal{L}^{-2}E_{n-1}\mathcal{V} = -4E_{n-1}.$$

Hence, $A^n_s(E_{n-1}, E_{n-1}) = 0$. Since $E_{n-1} \in \mathcal{V}$, $\nabla^s_{E_{n-1}} E_{n-1} = 0$ (as shown in [4, 9.3]). When f is holomorphic with conformal shape form, we thus see

$$\tau_{\mathcal{Q}}(\mu) = 0.$$

□

Example 5.2.6. When $f : M \to \mathbb{CP}^n$ is a holomorphic surface, it is necessarily minimal. Since M is 2-dimensional, $A_f(\xi)$ only has two eigenvalues $\pm\kappa$, and so it necessarily has conformal shape form as well. Therefore $\gamma : UM^\perp \to \mathcal{Q}$ is a minimal Lagrangian submanifold.

Example 5.2.7. If $P(X, Y, Z)$ is an irreducible homogeneous complex polynomial of degree 3, then the level set

$$M = \left\{[X, Y, Z] \in \mathbb{CP}^2 : P(X, Y, Z) = 0\right\}$$

is an *elliptic curve*. If M is non-singular (meaning $\nabla P \neq (0, 0, 0)$ when $P = 0$), then M is diffeomorphic to a 2-torus and is thus a Riemann surface. Since every compact Riemann surface admits a holomorphic embedding into complex projective space [20, 5.19], we therefore get a family of embedded minimal Lagrangian submanifolds of $\mathcal{Q}$.

5.3 Coisotropic submanifolds

5.3.1 The normal metric

We shall now consider the case where $f : M \to \mathbb{CP}^n$ is coisotropic, so $\mathcal{I}TM^\perp \subset TM$. This case will have some complications due to $\mathcal{C}_2 \cap \mathcal{H}_M$ being non-trivial. As usual, we begin by constructing a local adapated frame for the decomposition of $\mathcal{C}$ which is orthonormal with respect to h_n. To simplify calculations we would like this frame to also respect the decomposition $\mathcal{C}_1 \oplus \mathcal{C}_2$.

Given a point $(p, \xi) \in UM^\perp$, when M is coisotropic, TN splits into three components: $TM^\perp$, $\mathcal{I}TM^\perp$ and $TM \cap \mathcal{I}TM$ (with the latter component, the *holomorphic tangent space*, being trivial when M is Lagrangian). We start by choosing an orthonormal frame $V_{m+1}, ..., V_{n-1}, \xi$ for $T_pM^\perp$. From this we define vectors $V_i = \mathcal{I}V_{m+i}$ for $i \in \{1, ..., n - m - 1\}$. We then choose an orthonormal frame $\{V_{n-m}, ..., V_{m-1}\}$ for $T_pM \cap \mathcal{I}T_pM$ (when M is not Lagrangian). Finally we define $V_m = \mathcal{I}\xi$.

We define corresponding vectors in $T_\xi UM^\perp$ as

$$\tilde{V}_i = \bar{V}_i - J\overline{B_f(\xi)V_i},$$
$$\tilde{V}_\beta = -J\overline{V}_\beta.$$

Let $U \subset M, V \subset UM^\perp$ be a pair of open subsets such that $\pi^\perp(V) = U$. $V \subseteq UM^\perp$. Then for a smooth choice of $\tilde{V}_A : V \to TU \subset TN$, by Corollary 5.1.5 we can define a local adapted orthonormal frame for $\mathcal{H}_M \oplus \mathcal{V}_M$ as

$$E_i = \left\{ \frac{\tilde{V}_i}{||\tilde{V}_i||} : i = 1, ..., m \right\},$$
$$E_\beta = \left\{ \tilde{V}_\beta : \beta = m+1, ..., n-1 \right\}, , \tag{42}$$

where $B_f(\xi)$ is the adapted shape operator. We also define corresponding $W_i = d\pi_N(E_i) = ||W_i||V_i,\ W_\beta = d\pi_N(JE_\beta) = V_\beta$.

We shall turn again to the decomposition (34) for $\tau_n(\mu)$. By Proposition 5.1.6, $\tau(\mu)^i_2$ still vanishes for coisotropic submanifolds.

For the remaining terms, where before we were able to eliminate the terms of the form $\langle[\mathfrak{n}, \mathfrak{m}], \mathfrak{m}\rangle$ by the use of results such as Lemma 5.2.4, for coisotropic submanifolds this is not generally the case. To this end, for $j = 1, ..., m$ we define one-forms

$$m_j : TV \to \mathbb{R};\ X \mapsto \langle[\varphi_\mathfrak{n}(E_j), \varphi_\mathfrak{m}(E_j)], \varphi_\mathfrak{m}(JX)\rangle\,.$$

Following the arguments for the sphere and holomorphic case, we acquire:

$$\tau_n(\mu)^i = -\sum_{j=1}^m \left(g\left(\left(\nabla^f_{W_j} B_f(\xi) \right) W_j, W_i \right) - m_j(E_i) \right),$$
$$\tau_n(\mu)^\beta = \sum_{j=1}^m \left(g\left(\nabla^f_{W_j} df(W_j), W_\beta \right) + m_j(E_\beta) \right).$$

In order to relate $\tau_n(\mu)^i$ to $\tau(f)$, we consider

$$g\left(\left(\nabla^f_{W_j} B_f(\xi) \right) W_j, W_i \right) = g\left(\nabla_{W_j} \left(A_f(\xi)\mathcal{L}W_j \right) - A_f\mathcal{L}(\xi)(\nabla^f_{W_j} W_j)^\top, W_i \right).$$

Since we've chosen our frame to respect $\mathcal{C}_1 \oplus \mathcal{C}_2$, this becomes

$$\begin{aligned} g\left(\left(\nabla^f_{W_j} B_f(\xi)\right) W_j, W_i\right) &= g\left(\mathcal{L}\left(\nabla^f_{W_j} A_f(\xi)\right) W_j, W_i\right) \\ &= g\left(\left(\nabla^f_{W_j} \mathrm{I\!I}_f\right)(W_j, \mathcal{L}W_i), \xi\right). \end{aligned}$$

To make use of the Codazzi equation, for a vector $Z \in T_p^{\perp} M$ we consider

$$g\left(R^N(W_j, \mathcal{L}W_i)W_j, Z\right) = -3g(\mathcal{I}W_j, \mathcal{L}W_i)g(\mathcal{I}W_j, Z).$$

Since M is coisotropic and by our choice of frame, either $\mathcal{I}W_j \in TM$ or $\mathcal{I}W_j \in T^{\perp}M$. In either case,

$$(R^N(W_j, \mathcal{L}W_i)W_j)^{\perp} = 0.$$

We thus have the following result

Lemma 5.3.1. *At a point $\xi \in UM^{\perp}$ with the local adapted frame described in (42) for a neighbourhood $V \subset UM^{\perp}$ about ξ:*

$$h_n(\tau_n(\mu), JE_i) = -\sum_{j=1}^{m} \left(g\left(\left(\nabla_{d\pi_N(\mathcal{L}E_j)} \mathrm{I\!I}_f\right)(d\pi_N(E_j), d\pi_N(E_j)), \xi\right) - m_j(E_i)\right)$$

$$h_n(\tau_n(\mu), JE_\beta) = \sum_{j=1}^{m} \left(g\left(\mathrm{I\!I}_f\left(d\pi_N(E_j), d\pi_N(E_j)\right), d\pi_N(JE_\beta)\right) + m_j(E_\beta)\right),$$

where

$$m_j(X)|_\xi = g\left(R^N(\xi, B_f(\xi)W_j), W_j, d\pi_N(JX)\right).$$

To relate $\tau_n(\mu)$ to $\tau(f)$, we shall again require horizontal conformality of $\pi^{\perp}$.

Lemma 5.3.2. *Given a coisotropic submanifold $f : M \to \mathbb{CP}^n$, $\pi^{\perp}$ is horizontally conformal if and only if M has conformal adapted shape operator, i.e. there exists $r : UM^{\perp} \to \mathbb{R}$ such that $g(B_f(\xi)X, B_f(\xi)Y) = r(\xi)^2 g(X, Y)$. The conformal factor for $\pi^{\perp}$ is $\frac{1}{1+\tilde{\kappa}^2}$, where $\pm\tilde{\kappa}$ are the only eigenvalues of $B_f(\xi)$.*

Proof. Let $X_1, ..., X_m$ be orthonormal eigenvectors for $B_f(\xi)$ with eigenvalues $\tilde{\kappa}_j$. We define corresponding orthonormal vectors $Y_1, ..., Y_m \in \mathcal{H}_M$ such that

$$Y_j = \frac{\bar{X}_j - \tilde{\kappa}_j J\bar{X}_j}{\sqrt{1 + \tilde{\kappa}_j^2}}.$$

Because π_N is Riemannian and J is an isometry,

$$\begin{aligned} g\left(d\pi_N(Y_i), d\pi_N(Y_j)\right) &= \delta_{ij} - \langle \varphi_{\mathfrak{n}}(Y_i), \varphi_{\mathfrak{n}}(Y_j)\rangle \\ &= \delta_{ij} - \tilde{\kappa}_i\tilde{\kappa}_j g(d\pi_N(Y_j), d\pi_N(Y_j)). \end{aligned}$$

Hence, if $\pi^\perp$ is horizontally conformal with conformal factor a^2:

$$\tilde{\kappa}_i\tilde{\kappa}_j\delta_{ij} = \frac{1-a^2}{a^2}\delta_{ij},$$

and so for each $i = 1, ..., m$, $\tilde{\kappa}_i^2 = \frac{1-a^2}{a^2} =: \tilde{\kappa}^2$. Thus $B_f(\xi)$ is conformal with conformal factor $r(\xi)^2 = \tilde{\kappa}^2$.

Conversely, if $B_f(\xi)$ is conformal,

$$\delta_{ij} = (1+r^2)g(d\pi_N(Y_i), d\pi_N(Y_j)),$$

and so $\pi^\perp$ is horizontally conformal with conformal factor $\frac{1}{1+r^2}$. □

We have thus proven the following result.

Theorem 5.3.3. *Let $f : M \to (\mathbb{CP}^n, g)$ be an isometrically immersed coisotropic submanifold where $\mathbb{CP}^n \cong SU(n+1)/K$ and $US^n \cong SU(n+1)/H$ are equipped with the normal metric. Let $Z, W \in T_\xi(UM^\perp)$ such that Z is $\pi^\perp$-horizontal and W is vertical. If $\pi^\perp$ is horizontally conformal (and thus f has conformal adapted shape form), then*

$$\begin{aligned} h_n(\tau_n(\mu), JZ)|_\xi &= -\frac{1}{1+r(\xi)^2}\left(g\left(\nabla^\perp_{d\pi_N(\mathcal{L}Z)}\tau(f), \xi\right) - tr_{f^*g}m(Z)\right), \\ h_n(\tau_n(\mu), JW)|_\xi &= \frac{1}{1+r(\xi)^2}\left(g(\tau(f), d\pi_N(JW)) + tr_{f^*g}m(W)\right), \end{aligned}$$

where $r(\xi)^2 = \frac{1}{dim(M)}B_f(\xi)^2$ and

$$m(Z) : TM \times TM \to \mathbb{R};\ X, Y \mapsto g\left(R^N(\xi, B_f(\xi)X)Y, d\pi_N(JZ)\right).$$

Corollary 5.3.4. *When $f : M \to (\mathbb{CP}^n, g)$*

- *is minimal for $codim(M) > 1$;*

- *has constant mean curvature for codim*$(M) = 1$

with conformal adapted shape form, then $\mu : UM^\perp \to (U\mathbb{CP}^n, h_n)$ *is minimal (and thus* $\gamma : UM^\perp \to \mathcal{Q}$ *is minimal) if and only if* $tr_{f^*g} m(X) = 0$ *for all* $X \in TUM^\perp$.

To find examples for which μ is minimal, we now want to consider when m_j vanishes.

When $X = E_\beta$, using (27):

$$\begin{aligned} m_j(E_\beta)|_\xi &= -g(\xi, \mathcal{I}W_j)g(B_f(\xi)W_j, \mathcal{I}W_\beta) - 2g(\xi, \mathcal{I}B_f(\xi)W_j)g(W_j, \mathcal{I}W_\beta) \\ &= g(V_m, W_j)g(B_f(\xi)W_j, \mathcal{I}W_\beta) + 2g(V_m, B_f(\xi)W_j)g(W_j, \mathcal{I}W_\beta) \\ &= \delta_{jm}||W_m||g(B_f(\xi)W_m, \mathcal{I}W_\beta) + 2\delta_{j(\beta-m)}||W_j||g(B_f(\xi)V_m, W_j) \end{aligned}$$

(where we have noted that $B_f(\xi)W_A = q_A A_f(\xi)W_A$ is self-adjoint with respect to our choice of frame).

For $X = E_i$:

$$\begin{aligned} m_j(E_i)|_\xi = &-g(\xi, \mathcal{I}W_j)g(B_f(\xi)W_j, B_f(\xi)W_i) + g(\xi, \mathcal{I}B_f(\xi)W_i)g(B_f(\xi)W_j, \mathcal{I}W_j) \\ &+ 2g(\xi, \mathcal{I}B_f(\xi)W_j)g(W_j, \mathcal{I}B_f(\xi)W_i) \\ = &-\delta_{jm}||W_m||g(\mathcal{I}B_f(\xi)^2W_m, W_i) - g(B_f(\xi)V_m, W_i)g(B_f(\xi)W_j, \mathcal{I}W_j) \\ &- 2g(B_f(\xi)V_m, W_j)g(\mathcal{I}W_j, B_f(\xi)W_i). \end{aligned}$$

If we were to impose the condition that $\mathcal{I}\xi$ is an eigenvector of $B_f(\xi)$ (and thus $A_f(\xi)$) with eigenvalue $\tilde{\kappa}_m$, then $m_j|_\xi$ reduces significantly. Since ξ is orthogonal to $W_j, \mathcal{I}W_\beta$

$$\begin{aligned} m_j(E_\beta)|_\xi &= 0, \\ m_j(E_i)|_\xi &= -\frac{2\tilde{\kappa}_m\delta_{im}}{\sqrt{4+\tilde{\kappa}_m^2}} g(B_f(\xi)W_j, \mathcal{I}W_j). \end{aligned}$$

Example 5.3.5. When $f : M \to \mathbb{CP}^n$ is totally geodesic, $\pi^\perp$ is Riemannian, $\mathcal{I}\xi$ is an eigenvector of $A_f(\xi) = 0$ and so $m_j(X) = 0$.

Example 5.3.6. In the case of a hypersurface, $TM \cap \mathcal{I}TM$ is the contact structure orthogonal to $\mathcal{I}\xi$, hence when $\mathcal{I}\xi$ is an eigenvector of $B_f(\xi)$ the eigenvectors orthogonal to $\mathcal{I}\xi$ lie in $TM \cap \mathcal{I}TM$. We can thus choose our $V_1, ..., V_m$ so that they are eigenvectors of $B_f(\xi)$, in which case $m_j(X) = 0$.

Example 5.3.7. When $f : M \to \mathbb{CP}^N$ is Lagrangian, $g(B_f(\xi)W_j, \mathcal{I}W_j) = 0$, and so when $\mathcal{I}\xi$ is an eigenvector of $A_f(\xi)$, $m_j(X) = 0$.

5.3.2 The lift of the Kähler-Einstein metric

As with the holomorphic case, we can use (30) to acquire a similar result for $\tau_{\mathcal{Q}}(\mu)$.

Theorem 5.3.8. *Let $f : M \to (\mathbb{CP}^n, g)$ be an isometrically immersed coisotropic submanifold such that $\mathbb{CP}^n \cong SU(n+1)/K$ is equipped with the normal metric and $U\mathbb{CP}^n \cong SU(n+1)/H$ is equipped with the lift of the Kähler-Einstein metric $h_{\mathcal{Q}}$. Let $Z, W \in T_\xi(UM^\perp)$ such that Z is $\pi^\perp$-horizontal and W is vertical. If f has conformal adapted shape form, then*

$$\begin{aligned} h_{\mathcal{Q}}(\tau_{\mathcal{Q}}(\mu), JZ)|\xi &= -\frac{1}{1+r(\xi)^2}\left(g\left(\nabla^\perp_{d\pi_N(Z)}\tau(f), \xi\right) - tr_{f^*g} m(\mathcal{L}^{-1}Z)\right) \\ &\quad - \frac{1}{2} h_{\mathcal{Q}}\left(I\!I_\mu(E_m, E_m), JZ\right), \\ h_{\mathcal{Q}}(\tau_{\mathcal{Q}}(\mu), JW)|_\xi &= \frac{1}{1+r(\xi)^2}\left(g(\tau(f), d\pi_N(JW)) + tr_{f^*g} m(W)\right) \\ &\quad - \frac{1}{2} h_{\mathcal{Q}}\left(I\!I_\mu(E_m, E_m), JW\right), \end{aligned}$$

where $r(\xi)^2 = \frac{1}{dim(M)} B_f(\xi)^2$, $E_m = \frac{\overline{\mathcal{I}\xi} - J\overline{B_f(\xi)\mathcal{I}\xi}}{||\overline{\mathcal{I}\xi} - J\overline{B_f(\xi)\mathcal{I}\xi}||}$ *and*

$$m(Z) : TM \times TM \to \mathbb{R};\ X, Y \mapsto g\left(R^N(\xi, B_f(\xi)X)Y, d\pi_N(JZ)\right).$$

Proof. Given an orthonormal frame $E_A, ..., E_A$ for (V, h_n) as in (42), we can define an orthonormal frame for $(V, h_{\mathcal{Q}})$ by $\tilde{E}_1, ..., \tilde{E}_{n-1}$ such that

$$F_A = \frac{E_A}{\sqrt{q_A}} = \begin{cases} E_A, & E_A \in \mathcal{C}_1 \\ \frac{E_A}{\sqrt{2}}, & E_A \in \mathcal{C}_2 \end{cases}$$

With respect to this frame

$$\begin{aligned} h_{\mathcal{Q}}(A^n_{\mathcal{Q}}(F_A, F_B), Z) &= -\frac{1}{2}\left\langle [\beta_H(F_A), \beta_H(\mathcal{L}^{-1}F_B)] + [\beta_H(F_B), \beta_H(\mathcal{L}^{-1}F_A)], \beta_H(Z)\right\rangle \\ &= \frac{q_A - q_B}{2}\left\langle [\beta_H(F_A), \beta_H(F_B)], \beta_H(Z)\right\rangle . \end{aligned}$$

Hence, $A^n_{\mathcal{Q}}(F_A, F_A) = 0$. Since h_n and $h_{\mathcal{Q}}$ share the same decomposition, $\mathrm{II}^n_\mu(F_A, F_A) = \mathrm{II}^{\mathcal{Q}}_\mu(F_A, F_A)$, and so:

$$\begin{aligned} h_{\mathcal{Q}}(\tau_{\mathcal{Q}}(\mu), JX) &= h_n(\tau_{\mathcal{Q}}(\mu), \mathcal{L}^{-1}X) \\ &= h_n\left(\sum_{A=1}^{n-1} \mathrm{II}_\mu(F_A, F_A), \mathcal{L}^{-1}JX\right) \\ &= h_n\left(\sum_{A=1}^{n-1} \frac{1}{q_A}\mathrm{II}_\mu(E_A, E_A), \mathcal{L}^{-1}JX\right) \\ &= h_n\left(\tau(\mu) - \frac{1}{2}\mathrm{II}_\mu(E_m, E_m), \mathcal{L}^{-1}JX\right). \end{aligned}$$

Unlike in the proof of Theorem 5.2.5, $\mathcal{I}\xi$ is not vertical and so we cannot use the Sasaki metric connection to eliminate the additional term. If we assume f has conformal shape form, the result then follows from Theorem 5.3.3. □

It is important to note that we have not assumed that $\pi^\perp$ is horizontally conformal for $h_{\mathcal{Q}}$. If we consider vectors $Z_i \in \mathcal{C}_i \cap \mathcal{H}_M$, then if f has conformal adapted shape form (and thus $\pi^\perp$ is horizontally conformal for h_n),

$$\begin{aligned} h_n(Z_i, Z_i) &= a^2 g(d\pi_N(Z_i), d\pi_N(Z_i)) \\ h_{\mathcal{Q}}(Z_i, Z_i) &= \frac{a^2}{q_i} g(d\pi_N(Z_i), d\pi_N(Z_i)). \end{aligned}$$

Hence when M is coisotropic, $\pi^\perp$ cannot be simultaneously horizontally conformal for both metrics.

Example 5.3.9. If we were to again assume that $\mathcal{I}\xi$ is an eigenvalue of the adapted shape operator, then by the decomposition in Lemma 2.5.4 we can calculate

$$[\varphi_{\mathfrak{h}}(E_m), \varphi_{\mathfrak{p}}(E_m)] \in [\mathfrak{h}, \mathfrak{q}_2] = \{0\}.$$

In this case,

$$h_{\mathcal{Q}}(\mathrm{II}_\mu(E_m, E_m), JE_A) = \frac{\delta_{mA}}{2}\left\langle E_m \varphi_{\mathfrak{p}}(E_m), \varphi_{\mathfrak{p}}(JE_m)\right\rangle .$$

Hence, the term vanishes if $(\nabla^{\mu}_{E_m} d\mu(E_m), JE_m) = 0$.

If we were to instead assume that $\pi^{\perp}$ is horizontally conformal with respect to $h_{\mathcal{Q}}$, we obtain a result which doesn't depend on $\mathrm{I\!I}_{\mu}$.

Theorem 5.3.10. *Let $f : M \to (\mathbb{CP}^n, g)$ be an isometrically immersed coisotropic submanifold such that $\mathbb{CP}^n \cong SU(n+1)/K$ is equipped with the normal metric and $U\mathbb{CP}^n \cong SU(n+1)/H$ is equipped with the lift of the Kähler-Einstein metric $h_{\mathcal{Q}}$. Let $Z, W \in T_{\xi}(UM^{\perp})$ such that Z is $\pi^{\perp}$-horizontal and W is vertical. If $\pi^{\perp}$ is horizontally conformal with conformal factor a^2, then*

$$h_{\mathcal{Q}}(\tau_{\mathcal{Q}}(\mu), JZ)|_{\xi} = -a^2 \left(g\left(\nabla^{\perp}_{d\pi_N(Z)} \tau(f), \xi\right) - tr_{f^*g} m(\mathcal{L}^{-1}Z)\right),$$
$$h_{\mathcal{Q}}(\tau_{\mathcal{Q}}(\mu), JW)|_{\xi} = -a^2 \left(g(\tau(f), d\pi_N(JW)) + tr_{f^*g} m(W)\right),$$

where

$$m(Z) : TM \times TM \to \mathbb{R};\ X, Y \mapsto g\left(R^N(\xi, B_f(\xi)X)Y, d\pi_N(JZ)\right).$$

Proof. As we established in the previous proof, with respect to the frame $F_A = q_A^{-\frac{1}{2}} E_A$:

$$h_{\mathcal{Q}}(\tau_{\mathcal{Q}}(\mu), JX) = \sum_{A=1}^{n-1} h_n(\mathrm{I\!I}_{\mu}(F_A, F_A), \mathcal{L}^{-1}JX).$$

By Lemma 5.3.1, we thus have

$$h_{\mathcal{Q}}(\tau_{\mathcal{Q}}(\mu), JF_i) = -\sum_{j=1}^{m} \left(g\left((\nabla_{d\pi_N(F_i)} \mathrm{I\!I}_f)(d\pi_N(F_j), d\pi_N(F_j)), \xi\right) - m_j(\mathcal{L}^{-1}F_i)\right)$$
$$h_{\mathcal{Q}}(\tau_{\mathcal{Q}}(\mu), JF_{\beta}) = \sum_{j=1}^{m} \left(g\left(\mathrm{I\!I}_f(d\pi_N(F_j), d\pi_N(F_j))\mathcal{L}^{-1} d\pi_N(JF_B)\right) + m_j(\mathcal{L}^{-1}F_{\beta})\right).$$

Hence, if $\pi^{\perp}$ is $h_{\mathcal{Q}}$ horizontally conformal (and thus $a^{-1}d\pi_N(F_j)$ is an orthonormal basis for T_pM), by noting that $F_{\beta} \in \mathcal{C}_1$ the result follows. $\square$

Corollary 5.3.11. *When $f : M \to (\mathbb{CP}^n, g)$*

- *is minimal for codim$(M) > 1$;*
- *has constant mean curvature for codim$(M) = 1$*

such that $\pi^\perp$ *is horizontally conformal with respect to* $h_{\mathcal{Q}}$*, then* $\mu : UM^\perp \to (U\mathbb{CP}^n, h_{\mathcal{Q}})$ *is minimal (and thus* $\gamma : UM^\perp \to \mathcal{Q}$ *is minimal Lagrangian) if and only if* $tr_{f^*g} m(X) = 0$ *for all* $X \in TUM^\perp$*.*

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